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A THEORETICAL STUDY OF NONLINEAR
TRANSVERSE COMBUSTION INSTABILITY
IN LIQUID PROPELLANT ROCKET MOTORS

Technical Report No. 732

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ABSTRACT

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The nonsteady flow conditions inside the subsonic section of a Laval nozzle have been considered in detail. The results of this analysis are used to derive a general Nonlinear Transverse Admittance Relation.

Crocco's time-lag hypothesis has been employed in the derivation of a concentrated combustion zone boundary condition. The latter together with the Nozzle Admittance Relations have been used in the investigation of the nonsteady irrotational flow conditions inside combustion chambers of liquid-propellant rocket engines. In this study the existence of three-dimensional continuous waves which are periodic in time and have amplitudes of finite size has been proven.

These solutions were expressed as power series in terms of an amplitude parameter ϵ . The first term of this series (i.e., first-order solution) was used to study the manner in which increasing the Mach number of the mean flow affected the linear stability limits. The results of this analysis showed that such an increase resulted in shifting of the unstable region, which is associated with the pure transverse acoustic mode, to higher values of the time-lag. In addition new unstable regions, which are associated with the mixed acoustic modes, appeared. Increasing the Mach number of the mean flow also resulted in a very small increase of the minimum value of the interaction index.

The first and second-order solutions were used to calculate the nonlinear pressure wave form. These theoretical predictions are shown to be in excellent qualitative agreement with available experimental results.

The stability of finite amplitude waves was analyzed and the possibility of "triggering" combustion instability in the case of three-dimensional oscillations has been proven theoretically. This possibility has been verified experimentally.

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NOMENCLATURE

Symbols which appear in more than one chapter are presented in the general list. All other symbols are presented under the listing of the chapter in which they appear. Because of the large number of variables which appear in this report, some symbols have been used to designate more than one quantity. These quantities are unrelated and it is hoped that this procedure will cause no confusion.

General

t	-	time
z	-	axial space dimension
r	-	radial space dimension
θ	-	tangential space dimension
ϕ	-	potential function for steady state velocity
ψ	=	$\frac{1}{2} \bar{p} \bar{q} r^2$, stream function of the steady flow
ω	-	frequency of oscillation
y	=	ωt
L	-	length
q	-	velocity vector
u	-	axial component of velocity
v	-	radial component of velocity
w	-	tangential component of velocity
p	-	pressure
ρ	-	density
s	-	entropy
T	-	temperature and period of oscillation
λ	-	amplification factor
\tilde{s}	=	$\omega - i\lambda$

$i = \sqrt{-1}$, imaginary unit.

c_p - specific heat at constant pressure.

c_v - specific heat at constant volume.

$\gamma = c_p / c_v$

c - speed of sound.

\mathcal{L} - linear ordinary differential operator.

\mathcal{F} - linear partial differential operator.

\mathcal{R} - operator used to express injector-end boundary condition.

$Y = \omega \tau$

F - see Equation (II-51)

$\mathfrak{F} = \frac{u}{\bar{q}}$

$\eta = \frac{v}{r \bar{p} \bar{q}}$

$\mathfrak{S} = rw$

$\pi = \frac{p}{\gamma \bar{p}}$

$\kappa = \frac{\rho}{\bar{\rho}}$

$S = s$

$\Phi(\varphi)$, $U(\varphi)$, $V(\varphi)$, $W(\varphi)$, $R(\varphi)$ and $P(\varphi)$ - coefficients which respectively describe the axial dependence of F , \mathfrak{F} , η , \mathfrak{S} , κ and π .

$\Psi(\psi) = J_{n\nu}(S_{(n\nu,q)} \sqrt{\frac{\psi}{\psi_w}})$

$\mathcal{H}(\theta) = \cos n\theta$ or $\sin n\theta$ for the case of standing modes and $e^{\pm in\theta}$ in the case of travelling mode.

$K^{(1)} = \mathcal{H}(\theta) \Psi(\psi) e^{imy}$

ϵ - small expansion parameter and amplitude of the first order solution.

\underline{e} - unit vector

$A_{(n\nu,q)}$, $B_{(n\nu,q)}$, $C_{(n\nu,q)}$, $A_{(n\nu,q')}^{(j,q)}$, $B_{(n\nu,q')}^{(j,q)}$, $C_{(n\nu,q')}^{(j,q)}$,

$N_{(n\nu,q')}^{(\nu,\nu)}$, $D_{(n\nu,q')}^{(\nu,\nu)}$, $M_{(n\nu,q')}^{(\nu,\nu)}$ - constants which are defined in Appendix C.

σ - axial component of the entropy evaluated at the origin.

$S_{(n\nu,q)}$ - root of $\frac{d}{dx} J_n(x) = 0$.

W , g and ℓ - see Equation (III-36)

$\sigma(\)$ - order of $(\)$.

$\chi, \zeta, \lambda, \tau$ - Hebrew letters used to represent the inhomogeneous parts of the higher order equations in the case of irrotational flow.

$$\mu = \frac{1}{\underline{x}} \frac{d}{dx} \underline{\Phi}$$

Γ - see Equation (II-185b)

n - interaction index

τ - time-lag

A - represents the inhomogeneous part of the axial-momentum equation.

B - represents the inhomogeneous part of the radial-momentum equation.

C - represents the inhomogeneous part of the transverse-momentum equation.

D - represents the inhomogeneous part of the entropy equation.

E - represents the inhomogeneous part of the continuity equation.

G - represents the inhomogeneous part of the Equation of State.

I - used to represent the inhomogeneous part of the differential equation for Φ .

C - represents either a constant of integration or constants which appear in the inhomogeneous part of various differential equations.

Δ - difference operator.

∇ - gradient.

THET = $\bar{q} \cdot L_{c.c.}$

SNH = $S_{(\nu,h)}$

Subscripts

- (k_m, n, v, q) - denotes a particular eigenfunction; k_m is the coefficient of y , n is the coefficient of θ and q identifies the root of the Bessel function which is being considered.
- i - identifies the imaginary part of a solution.
- r - identifies the real part of a given quantity.
- e - entropy or nozzle entrance
- v - vorticity
- n - nonlinear
- $\phi, \psi, \theta, \gamma$ - indicate a given direction or differentiation with respect to the particular variable.
- x - represents either e , v or n
- w - wall
- s - standing
- T - travelling
- h - homogeneous
- $c.c.$ - combustion chamber
- b - burning
- i - injection
- p - particular solution
- \rightarrow - vectorial quantity

Superscripts

- $0, 1, 2, 3$ - indicate order of the particular expression.
- j - indicates order of the particular expression.
- \wedge - denotes a dimensional quantity.
- \sim - with the exception of n and γ this symbol denotes quantities that describe conditions inside the combustion chamber.
- $*$ - complex conjugate of a quantity.
- $-$ - denotes a quantity which describes steady-state conditions.
- $'$ - total differentiation with respect to independent variable.

Special Nomenclature for Chapter IV

- \mathcal{C} - see Equation (III-7).
- $W = \frac{c_3^{(1)}}{c_2^{(1)}}$
- H - see Equation (IV-11).
- \mathfrak{J} - see Equation (IV-9a).
- T, ∇, h and g - see Equations (IV-25) through (IV-29d).
- X - see Equation (IV-35).
- \mathcal{A} and \mathcal{B} - see Equations (IV-40a) and (IV-40b).
- Φ - a particular solution, part of the general solution for Φ .
- $/A$ - vector defined in Equation (IV-60).
- $f(n, \omega, Y...)$ - see Equation (IV-8a) on page 171.
- $g(n, Y...) = f_r(n, \omega, Y...)$.
- \mathcal{D} - defined in Equation (IV-68).
- $\tilde{n}, \tilde{\epsilon}, \tilde{A}, \tilde{B}, \tilde{Q}_1$, and \tilde{C}_1 - defined in Equation (IV-76).
- \mathfrak{z} - potential function for the steady-state velocity.

Special Nomenclature for Chapter V

- T - period of oscillation.
- S - area.
- $y = \frac{v}{p}$, Nondimensional Admittance Relation.
- \hat{n} - unit normal vector.
- \hat{K} - proportionality constant.

Subscripts for Chapter V

- N - nozzle.
- $I.E.$ - injector-end.

Special Nomenclature for Chapter II

- L - see Equation (II-44g).
 F - arbitrary function.
 f_0 - see Equation (II-57).
 f_1 - see Equation (II-77).
 f_2 - see Equation (II-79).
 f_3 - see Equation (II-81).
 H - see Equation (II-60).
 \tilde{H} - see Equation (II-119).
 M - see Equation (E-5) of Appendix E.
 K - see Equation (II-181).
 \hat{z} - see Equation (II-183).
 $\bar{x} = \bar{c}^2 f_0 \Phi_h \Gamma_x$
 $\beta(\varphi)$ - see Equation (II-198).

Subscripts for Chapter II

- r - reference quantity.

Special Nomenclature for Chapter III

- \dot{m} - rate of gas generation per unit time per unit area.
 E - hypothetical quantity representing the amount of necessary conditioning prior to chemical reaction.
 f - rate of accumulation of E .
 Q - represents rate of gas generation per unit time volume and the inhomogeneous parts of injector face boundary condition.
 $[]$ - operator defined in Equation (III-48).
 L^* - characteristic length.

CHAPTER I.

INTRODUCTION

Nature of the Problem

The operation of liquid propellant rocket engines is never perfectly smooth. Experimental measurements of any of the physical factors, which describe the conditions inside the combustion chambers of these engines (i.e., pressure, temperature, etc.), indicate that the latter oscillate with time. These oscillations may be described as being of a nondestructive or a destructive nature. In the first case the oscillations are caused by some fluid mechanical effects (e.g., shear flow) and the correlation between the fluctuations at two different locations, or two different instants, disappears as soon as the time or space interval which separates them is no longer small. Because of the random nature of these oscillations, the net contribution of their integrated effect over time or space is identically zero. Consequently their presence does not interfere with the operation of the engine. When this type of nonsteadiness is present[#] the combustion process, which under these conditions can be maintained without any difficulty, is said to be "rough". Another important observation to note is the fact that the size of the amplitudes of these random fluctuations has no effect upon the operation of the system as a whole.

In the case of destructive (or detrimental) oscillations definite correlation between the fluctuations at any two locations (at a given instant) or at any two instants (at a given location), has been observed. On a pressure recording (see Figure 1) these fluctuations usually appear as an "organized" wave motion which is characterized by a definite value of the frequency. The net contribution of the integrated effect of these fluctuations, over space or time, results in additional mechanical and thermal loads that the system must withstand. The additional mechanical load will unduly stress or fatigue both the chamber and its mounting causing mechanical failure. The heat transfer rate to the chamber walls will,

[#] A typical pressure recording of these fluctuations (which appears as "combustion noise") is given in Figure 1.

under certain conditions,[#] increase several fold and result in a rapid deterioration and burnout of the combustion chamber walls. Finally, if the rocket engine can withstand both of these effects, then secondary oscillations can be set up in its delicate control and guidance system and thus destroy its effectiveness.

The presence of this type of oscillation is usually associated with a nonsteady combustion process. The organized nature of these oscillations suggests that some kind (or kinds) of feedback mechanism (or mechanisms) must exist between the combustion process and the wave system. Through this system the energy, which is necessary for maintaining the oscillations, is continuously being fed from the combustion process into the wave system. The thorough understanding of this mechanism is of utmost importance if the problems associated with unstable operation of rocket engines (and which hinder the development of the science of rocketry) are ever to be solved.^{##}

Depending on the frequency of the oscillations the problems associated with combustion instability are usually divided into the following three groups:^{###}

1.) Low frequency combustion instability which is characterized by wave oscillations whose frequencies range from 10 to 200 cycles per second. This type of unstable combustion was the first one to be observed and investigated.^{####} It is less detrimental than the other types of combustion instability and the mechanism responsible for its appearance as well as the means to avoid it are presently believed to be known. This type of instability will not be considered in the present analysis.

[#] This will occur in the case of high frequency oscillations.

^{##} In spite of continuous progress in this field the instability problems associated with the development of different rocket engines must still receive separate attention. Up to date there is no theory which is general enough and which can be applied with confidence to the design of stable rocket engines.

^{###} It is also believed that the basic mechanisms, which are responsible for the maintenance of these oscillations, are different at various frequency regimes.

^{####} See References 1, 15, 16 and 17 for full discussion of the problems associated with this type of combustion instability. The interaction between the oscillations inside the combustion chamber and the feedline is believed to be the mechanism responsible for the maintenance of these oscillations.

2.) The second known type of combustion instability is characterized by wave oscillations whose frequency is several hundred cycles per second. The presence of this type of oscillation, which is less frequently observed, may be attributed to the appearance of entropy waves inside the combustion chamber. A complete discussion of this phenomenon can be found in References 1 and 12.

3.) The third known type of combustion instability and the one that will be investigated in this thesis is best known as high frequency combustion instability. It represents the case of forced oscillations of the combustion chamber gases which are driven by the combustion process and interact with the resonance properties of the chamber geometry. Experiments with unstable rocket engines indicate that the oscillations which occur inside the combustion chamber may be longitudinal or transverse.[#] These latter are similar to the well known acoustic modes and are described through the use of the same nomenclature. These oscillations have frequencies which fall within a few percent of the frequencies of those acoustic modes which would appear in a closed-end container having the same geometry as the particular combustion chamber whose unstable behavior is being considered. In the case of transverse oscillations (which will be discussed in great detail in the following analysis) the following are, in the order of increasing frequencies, the most frequently observed modes of instability: first tangential, second tangential, first radial and the first mixed radial-tangential mode. Transverse waves can appear either in a standing wave pattern, in which the nodal surfaces remain fixed in space or in the form of a spinning wave in which case the nodal surfaces rotate, in either the clockwise or counterclockwise directions, with the frequency of the oscillation. Because of its frequent appearances and highly destructive nature, the phenomenon of high frequency combustion instability has been in the past fifteen years, and probably will be in the near future, the subject of considerable research effort. A summary of the major contributions in this field will be presented in the next section.

[#] Provided that the necessary conditions for instability are met, pure transverse oscillations will appear when the chamber length to diameter ratio is much smaller than one (i.e., $\frac{L}{D} \ll 1$); longitudinal oscillations will occur when this ratio is much larger than one (i.e., $\frac{L}{D} \gg 1$) while combined longitudinal-transverse oscillations will occur when $\frac{L}{D} \sim 1$.

Previous Work in the Field

It was the need for larger power plants which could carry heavier payloads into space, that brought the problems associated with high frequency combustion instability to the attention of scientists. Since the early 1950's, considerable research effort, both experimentally and analytically, has been devoted towards improving the understanding of the basic mechanisms which control the behavior of high frequency combustion instability. Crocco¹⁸, in the first comprehensive paper on the subject, suggested that the existence of a time-dependent sensitive time-lag[#] is the coordinating mechanism which is responsible for the appearance of these self sustained oscillations. At the time of its publication the predictions of this paper were, at least qualitatively, in agreement with the experimentally observed behavior of unstable rocket engines. Crocco's original paper¹⁸, and subsequent work by Crocco and Cheng, as well as any other relevant work in the field of combustion instability which was available at the time, are presented in a comprehensive monograph on the subject published by Crocco and Cheng in 1956. The theory and results presented in this monograph are limited, however, to the case of small amplitude longitudinal oscillations.

Using an approach which was analogous to the one used by Crocco and Cheng, Scala¹² employed a pressure sensitive time-lag to study the case of transverse combustion instability. Scala's predictions disagreed, however, with the experimentally observed fact that the stability limits of transverse modes are dependent on their form (i.e., spinning or standing waves). To account for this discrepancy, Reardon² extended Scala's work by incorporating velocity-effects into the analysis. These are supposed to account for the transverse motion of the vaporized propellants, oxidants and combustion products which may periodically vary the relative local concentration of the reacting agents and thus modify the stability

The use of a constant time-lag in the solution of rocket propulsion problems was originally suggested by von Karman in 1942. This concept was later employed by various investigators studying the problems associated with low frequency combustion instability. Crocco¹⁸ was the first one to argue that the overall time-lag can be considered as a summation of two parts; i.e., $\tau(t) = \tau_i + \tau_s(t)$ where τ_i is the insensitive time-lag, which is a constant quantity while $\tau_s(t)$ is the time-dependent sensitive time-lag. The oscillatory nature of $\tau_s(t)$ was shown to be the controlling mechanism which could be responsible for the maintenance of the high frequency oscillations.

characteristics of the engine. Culick⁷ preferred to use an energy source rather than a mass source (which was used in the formulation of the time-lag theory) as a forcing function which is responsible for the appearance of instability. Culick's formulation and conclusions are applicable to both liquid and gas rockets. The theories presented in all of the above mentioned references were limited to the case of small oscillations and their predictions and conclusions should be applied with caution.

In studies of rocket instability the experimental verification of linear theories is a very difficult task. The majority of the available experimental information has been obtained in the nonlinear regime; and thus making a direct check of the theory is practically impossible. Indirect comparisons, reported in Reference 1, showed good qualitative agreement between linear theory and experiments. In a later publication,¹⁹ Crocco, Grey and Harrje report excellent quantitative and qualitative agreement between the linear theory and a set of experiments (in which a variable length rocket engine operated over a wide range of mixture ratios was used) specifically designed for testing this theory. Reardon, in Reference 2, reports about a separate set of experiments in which a sector motor was used for the purpose of correlating theory and experiments in order to "estimate" the values of the parameters (i.e., τ and n) which determine the transverse linear stability limits.

Frequently during the course of these experiments different phenomena, which could not be explained by means of linear theories, have been observed. As examples of observed nonlinear phenomenon, we can mention the instability which is triggered by disturbances associated with the transient behavior of the engine immediately after the start of the operation. This kind of instability can be eliminated by attaching a diametral baffle to the injector face. Experiments with these baffles² also show that their presence may introduce instabilities of a different nature, that is, operating the same engine under similar conditions with and without a diametral baffle (which is burned and thus disappears within a period of one second) will result in the appearance of different modes of instability.

These observations together with the difficulties associated with the precise experimental determination of the linear stability limits point out the necessity for developing a nonlinear theory which could account for the observed phenomena and be directly compared with experimental findings.

Linear vs. Nonlinear Instabilities

The phenomenon of combustion instability is considerably nonlinear. The nonlinearities of the problem can be attributed to: #

a) The nonlinearity of the conservation equations which describe the flow conditions inside the combustion chamber and which control the behavior of the various thermodynamic variables.

b) The nonlinearity of the mechanism of instability. As examples we can mention the Arrhenius expression for the chemical reaction rate which states that the reaction rate is exponentially related to the temperature. Another example is the phenomenon of droplet shattering which becomes important only at a certain level of the amplitude of the oscillation and is not linearly related to it.

c) The third type of nonlinearity is of a purely kinematic nature. This phenomenon refers to the fact that when a transverse component of the velocity is very small, then the velocity vector is linearly independent of its presence; that is, if $\frac{v}{u} \ll 1$ (where u and v are the velocity components, which in rectangular coordinates are respectively parallel to the x and y axis) then

$$|q| = \sqrt{u^2 + v^2} \sim u(1 + \frac{1}{2}(\frac{v^2}{u^2})) \text{ is linearly dependent of } v \text{ while if}$$

$\frac{v}{u} \gg 1$ then $|q| = \sqrt{u^2 + v^2} \sim v(1 + \frac{1}{2}(\frac{u^2}{v^2}))$ and q is linearly dependent on v .

Fully established combustion instability can be obtained by either a spontaneous growth of a small disturbance in the case of linearly unstable engines or by "triggering" action. When the latter possibility exists,

A more detailed discussion of the nonlinear phenomenon as well as a complete theoretical review of the instability problem can be found in Reference 6.

disturbances with amplitude beyond a certain level may induce instability in an otherwise linearly stable system. While the conclusions of linear analysis may be sufficient to avoid spontaneous instability, this may not be the case when "triggering" is being considered. The latter depends on the presence of finite amplitude disturbances whose investigation requires the solution of the full nonlinear equations.

Consideration of the instability problem through the use of the full nonlinear equations and boundary conditions is expected to result in a) the modification of stability boundaries which were established by means of linear theories and b) the "discovery" of previously unknown nonlinearly unstable regions. These possibilities may be best explained by means of the schematic diagram presented in Figure 2[#]. The unstable region on the left of this figure represents the first possibility mentioned above; in which case the nonlinear effects resulted in the modifications of linear stability limits. This difficulty can be avoided in practice by designing engines which operate in a region sufficiently removed from the linear stability limits (e.g., at mixture ratios between C and D). Thus information provided by linear theories may be sufficient to avoid this difficulty and the extension of the linear theory into the nonlinear regime may not be necessary. Unfortunately, experiments seem to indicate that unstable regions like the one shown on the right of Figure 2 are also in existence. In this case a finite amplitude disturbance can introduce instability in regions (in this example we refer to ranges of the mixture ratio) which are linearly stable. Thus for mixture ratios between D and E every small amplitude disturbance will decay while disturbance with amplitudes larger than certain threshold values, which are indicated by the lower half of the boundary surrounding the unstable region, induce instability. Since the operation of rocket engines is never perfectly smooth, an apriori knowledge of the location of these unstable regions is of utmost importance for the design of unconditionally stable engines. The importance of having such a design criterion provides sufficient motivation for the study of the nonlinear stability problem.

Because of the complexity of the theoretical treatment of nonlinear problems the number of available relevant publications in this field is relatively scarce. Among these we should mention the work of Maslen and

[#] The following discussion is qualitative in nature and the cases described in Figure 2 do not necessarily refer to the same rocket engine.

Moore.³ In this investigation the behavior of finite-amplitude transverse periodic waves was studied in detail. This work was limited, however, to fluid mechanical effects only. The effects of the combustion process, the steady-state or mean flow (which results from the presence of a combustion process) and the presence of a nozzle at one end of the combustion chamber, which are so important in the determination of the stability limits of rocket engines, were not considered at all. In conclusion Maslen and Moore show that contrary to the case of one-dimensional oscillations, in which a train of waves coalesces into a shock wave, finite-amplitude transverse periodic waves can be continuous. The continuous diffraction of the waves at the curved walls is suggested as a possible mechanism which is responsible for avoiding the steepening of these waves into a shock wave. Allen²⁰ studying the cold flow conditions in a simulated rocket motor and Reardon² taking measurements in an actual firing of a rocket motor report the observations of pressure waveforms having similar shape to the theoretically-predicted waveforms of Maslen and Moore.

Priem, in Reference 21, considers the nonlinear (stability) problem using an entirely different approach. In this case, the conservation equations which describe the nonsteady flow conditions inside an annular section of the combustion chamber of a liquid-propellant rocket engine were numerically integrated with time. An arbitrary disturbance was introduced into the initial conditions and its growth or decay with time (which was determined by the results of the numerical integration) determined whether the system was stable or unstable. This procedure was repeated several times in order to investigate the effect of different parameters (e.g., the form of the forcing function incorporated in the conservation equations, the Mach number inside the combustion chamber, etc.) on the stability of the engine. This approach is not, however, general enough and the calculated results depend on the particular perturbation that is introduced in the initial conditions.

The successful use of the time-lag concept in linear theories prompted Sirignano⁴ to consider its applicability to the study of the nonlinear problems. Assuming the existence of a concentrated combustion zone and considering the case of short nozzle (which in the case of

longitudinal oscillations permits the use of the assumption that Mach number at the nozzle entrance is approximately constant) Sirignano found the solutions for finite-amplitude, continuous, longitudinal periodic waves which could be either stable or unstable. The results of a specific numerical example, which was solved in this work, show that in the range of the parameters which are of physical interest the finite amplitude periodic waves are unstable; it was thus concluded that in the case of longitudinal instability triggering is possible and shock waves moving back and forth along the combustion chamber would most probably result.

An experimental program oriented towards the determination of the nonlinear stability limit has been underway at Princeton University for the past few years. In a series of tests the change of the stability limits of a given rocket engine, operating under unchanged specified conditions, due to the introduction of a series of disturbances of various amplitude levels is investigated. These tests are repeated for the purpose of investigating the effect of various parameters (i.e., injector orientation, injection element spacing, thrust level, etc.) on the nonlinear stability limits. In addition the effect of the transients that are present at the start of the operation of a rocket engine has been tested by introducing destructible diametral baffles at the injector face. Detailed descriptions of these tests can be found in References 22 and 23.[#]

Objective of Present Investigation

It is the purpose of the present analysis to develop the theory beyond the existing works on transverse linear instability and the work done by Maslen and Moore. As far as linear theories are concerned, the objectives of this work are to extend the existing results to include the effects of finite-amplitude oscillations. The possibility of triggering combustion instability will also be investigated. From the nonlinear point of view, it is the purpose of this work to investigate the interaction between the fluid mechanical processes on one hand and the combustion process on the other. The effect of the existence of a mean

[#] These references include an additional list of works which are concerned with various experimental aspects of combustion instability.

steady flow as well as the presence of a nozzle at one end of the combustion chamber, upon the stability of the transverse waves, will also be considered.

The simplifications in the analysis of longitudinal waves which resulted from the assumption that the subsonic portion of the nozzle is considerably shorter than the length of the combustion chamber cannot be applied in the study of transverse oscillations. In the latter case the transverse dimensions of the nozzle are, at least in the section which is closer to the combustion chamber, of the same order of magnitude as the corresponding dimensions in the combustion chamber. Consequently in order to investigate the stability of three-dimensional waves inside the combustion chamber, a nonlinear transverse admittance relation which represents the effect of the nozzle must be derived. The derivation of this relation is a problem by itself and is presented in Chapter II. In order to simplify the analysis of the problem the combustion process will be assumed to be concentrated within an infinitesimally thin zone immediately adjacent to the injector face. The derivation of the boundary condition which represents this condition is presented in Chapter III. In Chapter IV, the equations derived in the analysis of the nozzle flow are simplified to describe the flow conditions inside a cylindrical combustion chamber. The solutions of these equations will be obtained and the boundary conditions derived in Chapters II and III will be satisfied. The nonlinear pressure waveform as well as the stability of these waves will also be determined in Chapter IV. Specific numerical examples will be presented in Chapter V.

CHAPTER II.

DERIVATION OF THE TRANSVERSE NOZZLE ADMITTANCE RELATION FOR FINITE AMPLITUDE WAVES

Introduction

There are several fundamental differences between acoustical oscillations in a closed cylindrical chamber and the oscillations that may be present in the combustion chamber of a rocket engine. In the latter case, the oscillations are superimposed on a mean flow created by the combustion process. While in acoustics the amplitudes of the oscillations are small, this may not be the case with the oscillations that may occur in a combustion chamber. The waves generated by an oscillatory combustion process travel through the combustion chamber into the nozzle where they are partially transmitted and partially reflected. Consequently the closed-end boundary condition used in the classical acoustic problem must now be replaced by a different boundary-condition which is imposed by the presence of a converging-diverging nozzle at the end of the combustion chamber.[#]

When the nozzle is in supercritical operation, the flow downstream of the sonic point is supersonic and no finite-disturbance (i.e. finite-amplitude wave) can propagate through this point in the upstream direction. Thus the flow conditions downstream of the sonic point have no effect upon the conditions inside the combustion chamber. Consequently the point where the flow velocity becomes sonic, suggests itself as the natural location for prescribing the boundary condition for the combustion chamber flow. When the non-steady flow perturbations oscillate around the approximately-one-dimensional steady flow, the location of the sonic point will also oscillate around the nozzle throat (which represents the sonic point in the case of steady one-dimensional flow) alternately moving in the upstream and downstream directions. When this happens

[#] In the case of rocket engines the separation between the combustion chamber and the nozzle is artificial. It is usually introduced because of the simpler geometry and lower mean flow velocity that are characteristic of the combustion chamber and which considerably simplify the solutions of its flow problems.

the flow conditions at the nozzle throat also oscillate and are subsonic at one instant (when the sonic point is downstream of the throat) and supersonic at another. Consequently there exists a condition in which disturbances can periodically propagate from the nozzle throat in the upstream direction. At the instant when the flow at the nozzle throat is supersonic no disturbance, unless its amplitude there is infinite, can propagate in the upstream direction.[#] Mathematically, this situation can be expressed by requiring that the solutions for the flow field inside the converging section of the nozzle be always regular at the nozzle throat. If these solutions were allowed to be singular at the nozzle throat, a situation could exist in which finite-amplitude waves would continually propagate, through the nozzle throat, in the upstream direction. In view of what has been said so far the existence of such a situation is physically not possible. As will be shown, this regularity condition can be replaced by a complex relation between the pressure, entropy and velocity perturbations. This relation is expected to hold in any location along the converging portion of the nozzle and for three-dimensional perturbations has been termed The Nozzle Transverse Admittance Relation. Evaluated at the nozzle entrance, it forms the proper boundary condition for the oscillatory flow in the combustion chamber.

The problem of supercritical flow with oscillations in a converging-diverging nozzle was first treated by Tsien⁸ who considered the case in which the oscillation of the incoming flow is isothermal. The solution was found for both very low and very high frequencies of oscillations. Crocco, (in Reference 1, Appendix B), extended this study to include the nonisothermal case and covered the entire frequency range. Both of these investigations were limited to one-dimensional, or longitudinal, oscillations. In a later work Crocco⁵ extended his one-dimensional treatment of the problem to include the case of transverse oscillations.

[#] No continuous disturbance can travel upstream from the nozzle throat unless the amplitude of the disturbance at the throat is infinite. Shock waves are not considered in this analysis.

It is the purpose of this chapter to extend the above work, which is limited to small perturbations, to the case where non-steady perturbations about the one-dimensional mean flow have amplitudes of finite size. The Transverse Admittance Relation which is applicable to this case will be derived. This expression will provide an appropriate boundary condition for the case when the periodic flow oscillations inside the combustion chamber have amplitudes of finite size. The admittance relations corresponding to the isentropic or irrotational oscillations will be obtained as special cases of the general theory.

The work done in this chapter is concerned with the flow in an axi-symmetric nozzle. It could, however, be extended to the special case of two-dimensional nozzle (see Reference 5 for the linearized treatment of this case). This case will not be treated here. The admittance relation derived later in this chapter could be used as a boundary condition in the determination of the oscillatory flow conditions in any propulsive device which is followed by a converging-diverging nozzle operating in the supercritical range.

The use of the Transverse Nozzle Admittance Relation will be demonstrated later on in this thesis. In addition, the behavior and stability of finite-amplitude, transverse, periodic pressure waves inside the combustion chamber of liquid propellant rocket engines will be analyzed in a specific example.

Derivation of the Equations Describing the Nozzle Flow

The oscillatory conditions inside an axi-symmetric, slowly-convergent, subsonic portion of a nozzle operating in the supercritical range will be analyzed. The flow is assumed to be adiabatic and inviscid and to have no body forces and no chemical reactions. The fluid is assumed to be a perfect gas with constant specific heats. Using "hats" ^ to denote dimensional quantities and operators, the equations describing the gas motion can be written in the following form:

Continuity:

$$\frac{\partial \hat{p}}{\partial \hat{t}} + \hat{\nabla} \cdot (\hat{p} \hat{\mathbf{q}}) = 0$$

(II-1)

Momentum:

$$\frac{\partial \hat{\underline{q}}}{\partial \hat{t}} + \frac{1}{2} \hat{\nabla}(\hat{\underline{q}} \cdot \hat{\underline{q}}) + (\hat{\nabla} \times \hat{\underline{q}}) \times \hat{\underline{q}} + \frac{1}{\hat{\rho}} \hat{\nabla} \hat{p} = 0 \quad (\text{II-2})$$

where \hat{t} is the time, $\hat{\rho}$ and \hat{p} are the density and pressure and $\hat{\underline{q}}$ is the velocity vector.

Since the fluid is assumed to be inviscid and non heat-conducting, the energy equation in its simplest form expresses the constancy of the entropy of a fluid particle after it enters the nozzle. This is expressed by the following relation:

$$\frac{D\hat{S}}{D\hat{t}} = \frac{\partial \hat{S}}{\partial \hat{t}} + \hat{\underline{q}} \cdot \hat{\nabla} \hat{S} = 0 \quad (\text{II-3})$$

where

$$\hat{S} = \hat{c}_p \left(\frac{1}{\gamma} \ln \hat{p} - \ln \hat{\rho} \right) + \text{constant} \quad (\text{II-4})$$

is the entropy. In Equation (II-4) \hat{c}_p is the specific heat at constant pressure and γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. The equation of state for an ideal gas has been employed in the derivation of Equation (II-4). For later reference an expression for \hat{c} , the speed of sound in an ideal gas, will be given:

$$\hat{c} = \left(\gamma \frac{\hat{p}}{\hat{\rho}} \right)^{1/2} \quad (\text{II-5})$$

Let the subscript r indicate an unspecified reference quantity and \hat{L} a characteristic length, the following transformation will be used in the non-dimensionalization of Equations (II-1) through (II-5):

$$\underline{q} = \frac{\hat{\underline{q}}}{\hat{c}_r} \quad ; \quad P = \frac{\hat{p}}{\hat{p}_r} \quad ; \quad \rho = \frac{\hat{\rho}}{\hat{\rho}_r} \quad ; \quad C = \frac{\hat{c}}{\hat{c}_r} \quad (\text{II-6})$$

$$S = \frac{\hat{S}}{\hat{c}_r} \quad ; \quad \hat{L}_r \hat{\nabla} = \nabla \quad ; \quad t = \frac{\hat{c}_r \hat{t}}{\hat{L}_r} \quad ; \quad \omega = \hat{\omega} \frac{\hat{L}_r}{\hat{c}_r}$$

As will be shown later, the equations derived in this section could be used in the analysis of the combustion chamber flow as well as the nozzle flow. Consequently the choice of the reference quantities, which appear in Equation (II-6), will depend on the specific problem which is under consideration. In the present analysis the stagnation pressure, entropy, velocity of sound, etc. of the unperturbed, steady, one-dimensional flow will be used as the reference quantities. The radius of the nozzle throat, r_t , is used as the characteristic length.

Substituting the relations given in Equation (II-6) into Equations (II-1) through (II-5) yields the following non-dimensional form of these equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \quad (\text{II-7})$$

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2} \nabla (\mathbf{q} \cdot \mathbf{q}) + (\nabla \times \mathbf{q}) \times \mathbf{q} + \frac{1}{\rho} \nabla P = 0 \quad (\text{II-8})$$

$$\frac{\partial S}{\partial t} + \mathbf{q} \cdot \nabla S = 0 \quad (\text{II-9})$$

$$S = \frac{1}{\gamma} \ln P - \ln \rho + \text{constant} \quad (\text{II-10})$$

$$C = \left(\frac{P}{\rho} \right)^{1/2} \quad (\text{II-11})$$

It is well known that in the case of oscillatory motion with finite-amplitude waves, the frequency of the oscillations as well as the other "eigenvalues" appearing in the problem are amplitude dependent. In treating problems of this kind the following transformation

$$y = t\tilde{s} \quad (\text{II-12})$$

where

$$\tilde{s} = \omega - i\lambda \quad (\text{II-13})$$

becomes very useful. Here ω is the non-dimensional frequency and λ the amplification factor. In the investigation of periodic solutions λ is set equal to zero and the solutions of the problem are assumed to be proportional to $e^{im\tilde{s}t} = e^{imy} = e^{im\omega t}$ where m is a given integer. In the new time scale, $y = \omega t$, the period of the oscillations, T , is a known quantity (equal to $\frac{2\pi}{m}$)

but the transformation parameter \tilde{s} has yet to be determined. Assuming that $\lambda = 0$, then $\tilde{s} = \omega$ and the transformation of Equations (II-7), (II-8) and (II-9) into the new time scale yields the following set of equations:

$$\omega \frac{\partial \rho}{\partial y} + \nabla \cdot (\rho \underline{q}) = 0 \quad (\text{II-7a})$$

$$\omega \frac{\partial \underline{q}}{\partial y} + \frac{1}{2} \nabla (\underline{q} \cdot \underline{q}) + (\nabla \times \underline{q}) \times \underline{q} + \frac{1}{\rho} \nabla P = 0 \quad (\text{II-8a})$$

$$\omega \frac{\partial S}{\partial y} + \underline{q} \cdot \nabla S = 0 \quad (\text{II-9a})$$

The nonlinearity of the equations describing the flow conditions inside the nozzle suggests the use of perturbation techniques in their solutions. As is customary in such cases the dependent variables and eigenvalues which appear in the problem are assumed to have the following power series expansions:

$$\underline{q} = \bar{q} + \underline{q}''' \epsilon + \underline{q}^{(2)} \epsilon^2 + \underline{q}^{(3)} \epsilon^3 + o(\epsilon^4)$$

$$p = \bar{p} + p''' \epsilon + p^{(2)} \epsilon^2 + p^{(3)} \epsilon^3 + o(\epsilon^4)$$

$$\rho = \bar{\rho} + \rho''' \epsilon + \rho^{(2)} \epsilon^2 + \rho^{(3)} \epsilon^3 + o(\epsilon^4)$$

$$S = \bar{S} + S''' \epsilon + S^{(2)} \epsilon^2 + S^{(3)} \epsilon^3 + o(\epsilon^4)$$

$$\omega = \omega^{(0)} + \omega''' \epsilon + \omega^{(2)} \epsilon^2 + \omega^{(3)} \epsilon^3 + o(\epsilon^4) \quad (\text{II-14})$$

where ϵ is an amplitude parameter, whose definition will be given in another section of this work. Barred quantities represent the steady-state solutions. With the exception of the quantities representing the steady-state solutions, all the variables appearing in the series for p , ρ , s and \underline{q} are space (three dimensional) and time dependent. The steady-state quantities are assumed to be known and it is the purpose of this analysis to solve for the coefficients of the higher powers of ϵ . In what follows the power series given in Eq. (II-14) are substituted into Equations (II-7a), (II-8a), (II-9a) and (II-10) and the resulting equations are then separated according to powers of ϵ . This procedure reduces the original system of nonlinear partial differential equations into an infinite number of sets (systems) of linear partial differential equations. The solution of these equations (up to $o(\epsilon^3)$ [#]) is the objective of the analysis presented in this chapter.

It will be shown in Chapter IV that solutions up to third-order only are necessary for the approximate determination of the eigenvalue perturbations and the stability criterion of finite-amplitude periodic waves.

Following the procedure outlined in the previous section leads to the derivation of the following system of equations:

$$\nabla \cdot (\bar{\rho} \bar{\vec{q}}) = 0 \quad (\text{II-15a})$$

$$\frac{1}{2} \nabla (\bar{\vec{q}} \cdot \bar{\vec{q}}) + (\nabla \times \bar{\vec{q}}) \times \bar{\vec{q}} + \frac{1}{\delta \bar{\rho}} \nabla \bar{P} = 0 \quad (\text{II-15b})$$

$$\bar{\vec{q}} \cdot \nabla \bar{S} = 0 \quad (\text{II-15c})$$

$$\bar{S} = \frac{1}{\delta} \ln \bar{P} - \ln \bar{\rho} + \text{constant} \quad (\text{II-15d})$$

which describes the steady-state flow. When the steady-state flow is assumed to be one-dimensional (and hence irrotational), the above set of equations can be replaced by the following simpler set:

$$\nabla \cdot (\bar{\rho} \bar{\vec{q}}) = 0 \quad (\text{II-16a})$$

$$\frac{\bar{P}}{\bar{\rho}} = 1 - \frac{\delta-1}{2} \bar{\vec{q}} \cdot \bar{\vec{q}} \quad (\text{II-16b})$$

$$\bar{S} = \text{constant} \quad (\text{II-16c})$$

$$\bar{P} = \bar{\rho}^\gamma \quad (\text{II-16d})$$

Using the definition of the non-dimensional sonic velocity:

$$\bar{C} = \frac{\hat{\bar{C}}}{\hat{C}_0} = \left(\frac{\bar{P}}{\bar{\rho}} \right)^{1/2} \quad (\text{II-17})$$

together with Equations (II-16b) and (II-16d) gives

$$\bar{C}^2 = 1 - \frac{\gamma-1}{2} \bar{q}^2 = \bar{p}^{\gamma-1} \quad (\text{II-18})$$

Equation (II-18) shows that when the steady-state flow is irrotational the specification of the steady-state velocity distribution is equivalent to the specification of the mean density and pressure variations. The steady-state velocity distribution is determined by the nozzle geometry (or vice versa).

It will be appropriate at this point to introduce the steady-state stream and potential functions. When the nozzle is axi-symmetric, Equation (II-16a) can be used to define a stream function ψ :

$$r \bar{p} \bar{q} = \underline{e}_\theta \times \nabla \psi \quad (\text{II-19})$$

where r is the non-dimensional distance from the axis of symmetry and \underline{e}_θ is the unit vector in the tangential direction. Since the steady-state flow under consideration is irrotational, a potential function can also be defined:

$$\bar{q} = \nabla \phi \quad (\text{II-20})$$

The stream and potential functions introduced above are non-dimensional.

Having solved the equations describing the steady-state flow (assuming that the geometry of the convergent section of the nozzle is known) we now proceed to derive the systems of equations describing the behavior of the nonsteady perturbations. The first order system of equations is:

Continuity:

$$\omega^{(1)} \frac{\partial f^{(1)}}{\partial y} + \nabla \cdot (\bar{q} f^{(1)} + q^{(1)} \bar{p}) = 0 \quad (\text{II-21})$$

Momentum:

$$\omega'' \frac{\partial \underline{q}''}{\partial y} + \nabla(\underline{\bar{q}} \cdot \underline{q}''') + (\nabla \times \underline{q}''') \times \underline{\bar{q}} + \frac{1}{2} \frac{P'''}{\bar{P}} \nabla(\underline{\bar{q}} \cdot \underline{\bar{q}}) + \frac{1}{\delta \bar{P}} \nabla P'' = 0 \quad (\text{II-22})$$

Entropy:

$$\omega'' \frac{\partial S''}{\partial y} + \underline{\bar{q}} \cdot \nabla S''' = 0 \quad (\text{II-23})$$

Equation of state:

$$S'' - \frac{P''}{\delta \bar{P}} + \frac{P'''}{\bar{P}} = 0 \quad (\text{II-24})$$

Using the definition of \bar{c} , Equation (II-24) can be rewritten in the following form:

$$\bar{c}^2 S'' - \frac{P''}{\delta \bar{P}} + \bar{c}^2 \frac{P'''}{\bar{P}} = 0 \quad (\text{II-24a})$$

Substitution of the following relation

$$\begin{aligned} \nabla \left(\frac{P'''}{\delta \bar{P}} \right) &= \frac{1}{\delta \bar{P}} \nabla P''' - \frac{P'''}{\delta \bar{P}} \frac{\nabla \bar{P}}{\bar{P}} \\ &= \frac{1}{\delta \bar{P}} \nabla P''' - \frac{P'''}{\delta \bar{P}} \frac{\nabla \bar{P}}{\delta \bar{P}} \\ &= \frac{1}{\delta \bar{P}} \nabla P''' + \frac{P'''}{\delta \bar{P}} \frac{1}{2} \nabla \bar{q}^2 \end{aligned}$$

together with Equation (II-24a) into Equation (II-22) results in the following, more convenient, form of the momentum equation:

$$\omega^{(2)} \frac{\partial \underline{q}^{(2)}}{\partial y} + \nabla(\underline{\bar{q}} \cdot \underline{q}^{(2)}) + (\nabla \times \underline{q}^{(2)}) \times \underline{\bar{q}} + \nabla\left(\frac{p^{(2)}}{\rho}\right) - \frac{1}{2} S^{(2)} \nabla \bar{q}^2 = 0 \quad (\text{II-22a})$$

At this point it should be mentioned that the equations describing the steady-state and first order flows are identical to those used by Crocco⁵ in his derivation of the "linear", transverse nozzle admittance relation. Since the solution of these equations are necessary for the analysis of the higher order equations, they will be repeated in a later section of this chapter.

The second order system of equations is:

Continuity:

$$\omega^{(2)} \frac{\partial \rho^{(2)}}{\partial y} + \nabla \cdot (\underline{\bar{q}} \rho^{(2)} + \underline{q}^{(2)} \bar{\rho}) = - \nabla \cdot (\rho^{(1)} \underline{q}^{(1)}) - \omega^{(1)} \frac{\partial \rho^{(1)}}{\partial y} \quad (\text{II-25})$$

Momentum:

$$\begin{aligned} \omega^{(2)} \frac{\partial \underline{q}^{(2)}}{\partial y} + \nabla(\underline{\bar{q}} \cdot \underline{q}^{(2)}) + (\nabla \times \underline{q}^{(2)}) \times \underline{\bar{q}} + \frac{1}{2} \frac{\rho^{(2)}}{\rho} \nabla(\underline{\bar{q}} \cdot \underline{\bar{q}}) + \frac{1}{\rho} \nabla p^{(2)} \\ = - \left\{ \omega^{(1)} \frac{\partial \underline{q}^{(1)}}{\partial y} + \omega^{(1)} \frac{\rho^{(1)}}{\rho} \frac{\partial \underline{q}^{(1)}}{\partial y} + \frac{\rho^{(1)}}{\rho} \nabla(\underline{\bar{q}} \cdot \underline{q}^{(1)}) + \frac{\rho^{(1)}}{\rho} (\nabla \times \underline{q}^{(1)}) \times \underline{\bar{q}} \right. \\ \left. + \frac{1}{2} \nabla(\underline{q}^{(1)} \cdot \underline{q}^{(1)}) + (\nabla \times \underline{q}^{(1)}) \times \underline{q}^{(1)} \right\} \end{aligned} \quad (\text{II-26})$$

Entropy Equation:

$$\omega^{(2)} \frac{\partial S^{(2)}}{\partial y} + \underline{\bar{q}} \cdot \nabla S^{(2)} = - \underline{q}^{(1)} \cdot \nabla S^{(1)} - \omega^{(1)} \frac{\partial S^{(1)}}{\partial y} \quad (\text{II-27})$$

Equation of State:

$$S^{(2)} - \frac{P^{(2)}}{\delta \bar{P}} + \frac{P^{(2)}}{\bar{P}} = -\frac{1}{2} \left\{ \frac{1}{\delta} \left(\frac{P^{(2)}}{\bar{P}} \right)^2 - \left(\frac{P^{(2)}}{\bar{P}} \right)^2 \right\} \quad (\text{II-27})$$

Using the definition of \bar{c} , Equation (II-28) can be rewritten in the following form:

$$\bar{c}^2 S^{(2)} - \frac{P^{(2)}}{\delta \bar{P}} + \frac{P^{(2)}}{\bar{P}} \bar{c}^2 = -\frac{1}{2} \bar{c}^2 \left\{ \frac{\delta}{\bar{c}^4} \left(\frac{P^{(2)}}{\delta \bar{P}} \right)^2 - \left(\frac{P^{(2)}}{\bar{P}} \right)^2 \right\} \quad (\text{II-28a})$$

Using the following relation

$$\nabla \left(\frac{P^{(2)}}{\delta \bar{P}} \right) = \frac{1}{\delta \bar{P}} \nabla P^{(2)} + \frac{P^{(2)}}{\delta \bar{P}} \frac{1}{2} \nabla \bar{c}^2$$

together with Equation (II-28a) results in the following, more useful, form of the momentum equation:

$$\begin{aligned} \omega^{(2)} \frac{\partial \underline{q}^{(2)}}{\partial y} + \nabla(\underline{\bar{q}} \cdot \underline{q}^{(2)}) + (\nabla \times \underline{q}^{(2)}) \times \underline{\bar{q}} + \nabla \left(\frac{P^{(2)}}{\delta \bar{P}} \right) \\ - \frac{1}{2} S^{(2)} \nabla \bar{c}^2 = - \left\{ \omega^{(2)} \frac{\partial \underline{q}^{(2)}}{\partial y} + \omega^{(2)} \frac{P^{(2)}}{\bar{P}} \frac{\partial \underline{q}^{(2)}}{\partial y} + \frac{P^{(2)}}{\bar{P}} \nabla(\underline{\bar{q}} \cdot \underline{q}^{(2)}) \right. \\ \left. + \frac{P^{(2)}}{\bar{P}} (\nabla \times \underline{q}^{(2)}) \times \underline{\bar{q}} + \frac{1}{2} \nabla(\underline{q}^{(2)} \cdot \underline{q}^{(2)}) + (\nabla \times \underline{q}^{(2)}) \times \underline{q}^{(2)} \right\} \\ + \frac{1}{2} \left\{ \frac{\delta}{\bar{c}^4} \left(\frac{P^{(2)}}{\delta \bar{P}} \right)^2 - \left(\frac{P^{(2)}}{\bar{P}} \right)^2 \right\} \frac{1}{2} \nabla \bar{c}^2 \end{aligned}$$

(II-26a)

Using the equations derived so far and employing procedures similar to those used in the simplification of the first and second order equations leads to the derivation of the following system of third order equations:[#]

Continuity:

$$\omega^{(1)} \frac{\partial \rho^{(3)}}{\partial y} + \nabla \cdot (\bar{\rho} \underline{q}^{(3)} + \rho^{(3)} \bar{\underline{q}}) = - \nabla \cdot (\rho^{(1)} \underline{\underline{q}}^{(2)} + \rho^{(2)} \underline{\underline{q}}^{(1)}) - \omega^{(2)} \frac{\partial \rho^{(1)}}{\partial y} \quad (\text{II-29})$$

Momentum:

$$\begin{aligned} \omega^{(1)} \frac{\partial \underline{q}^{(3)}}{\partial y} + \nabla (\bar{\underline{q}} \cdot \underline{q}^{(3)}) + (\nabla \times \underline{q}^{(3)}) \times \bar{\underline{q}} + \nabla \left(\frac{\rho^{(3)}}{\bar{\rho}} \right) - \frac{1}{2} S^{(3)} \nabla \bar{q}^2 \\ = \left\{ \frac{\gamma}{\bar{c}^2} \left(\frac{\rho^{(1)}}{\bar{\rho}} \right) \left(\frac{\rho^{(2)}}{\bar{\rho}} \right) - \frac{\gamma^2}{3 \bar{c}^2} \left(\frac{\rho^{(1)}}{\bar{\rho}} \right)^3 - \left(\frac{\rho^{(1)}}{\bar{\rho}} \right) \left(\frac{\rho^{(2)}}{\bar{\rho}} \right) + \frac{1}{3} \left(\frac{\rho^{(1)}}{\bar{\rho}} \right)^3 \right\} \frac{1}{2} \nabla \bar{q}^2 \\ + \frac{\rho^{(1)}}{\bar{\rho}} \left\{ \nabla \left(\frac{\rho^{(2)}}{\bar{\rho}} \right) - \left(S^{(2)} + \frac{1}{2} \left(\frac{\gamma}{\bar{c}^2} \left(\frac{\rho^{(1)}}{\bar{\rho}} \right)^2 - \left(\frac{\rho^{(1)}}{\bar{\rho}} \right)^2 \right) \frac{1}{2} \nabla \bar{q}^2 \right\} \\ - \left(\frac{\rho^{(2)}}{\bar{\rho}} - \left(\frac{\rho^{(1)}}{\bar{\rho}} \right)^2 \right) \left(\frac{1}{2} S^{(1)} \nabla \bar{q}^2 - \nabla \left(\frac{\rho^{(1)}}{\bar{\rho}} \right) \right) - \nabla (\underline{q}^{(1)} \cdot \underline{q}^{(2)}) - \omega^{(2)} \frac{\partial \underline{q}^{(1)}}{\partial y} \end{aligned} \quad (\text{II-30})$$

Entropy Equation:

$$\omega^{(1)} \frac{\partial S^{(3)}}{\partial y} + \bar{\underline{q}} \cdot \nabla S^{(3)} = - \left\{ \bar{\underline{q}}^{(1)} \cdot \nabla S^{(2)} + \underline{q}^{(2)} \cdot \nabla S^{(1)} \right\} - \omega^{(2)} \frac{\partial S^{(1)}}{\partial y} \quad (\text{II-31})$$

Equation of State:

$$\begin{aligned} \bar{c}^2 S^{(3)} - \frac{\rho^{(3)}}{\bar{\rho}} + \bar{c}^2 \frac{\rho^{(1)}}{\bar{\rho}} = \frac{\gamma^2}{3 \bar{c}^2} \left(\frac{\rho^{(1)}}{\bar{\rho}} \right)^3 - \frac{\gamma}{\bar{c}^2} \left(\frac{\rho^{(1)}}{\bar{\rho}} \right) \left(\frac{\rho^{(2)}}{\bar{\rho}} \right) \\ - \frac{1}{3} \left(\frac{\rho^{(1)}}{\bar{\rho}} \right)^3 + \left(\frac{\rho^{(1)}}{\bar{\rho}} \right) \left(\frac{\rho^{(2)}}{\bar{\rho}} \right) \end{aligned} \quad (\text{II-32})$$

[#] In Chapter IV it will be shown that $\omega^{(1)} = 0$. Consequently terms proportional to $\omega^{(1)}$ will be omitted from the third order equations.

Choice of Coordinate System

Before proceeding further in the analysis, a coordinate system, appropriate to the introduction of the boundary condition at the nozzle walls, must be chosen. Culick⁷, who worked on a similar problem, developed his equations in an unorthogonal coordinate system. In this work an approach similar to the one used by Crocco⁵ will be used. When treating an axi-symmetric nozzle, it is convenient to let the steady-state potential function ϕ replace the axial variable, and the steady-state stream function ψ replace the radial variable. Letting δs and δn be respectively elementary (non-dimensional) lengths in the direction of the unperturbed streamlines and of their normals on the meridional planes (see Figure 3) Equations (II-19) and (II-20) can then be written in the following form:

$$\bar{q} = \frac{d\phi}{\delta s} \qquad r\bar{p}\bar{q} = \frac{d\psi}{\delta n} \qquad \text{(II-33)}$$

The third independent variable, θ , indicates the azimuthal variation. The detailed derivation of the expressions for the divergence, gradient and rotor in a (ϕ, ψ, θ) coordinate system is given in Appendix D.

Solution of the Equations

Assuming that the walls of the nozzle under consideration are slowly convergent (which is consistent with the assumption that the unperturbed flow is one-dimensional), the obliquity of the streamlines with respect to the axis of symmetry is sufficiently small so that its cosine is practically one and the element of normal δn along the surface $\phi = \text{constant}$ can be identified with dr . Using this assumption in Equation (II-33) gives

$$\psi = \frac{1}{2} \bar{p}(\phi) \bar{q}(\phi) r^2 \qquad \text{(II-34)}$$

Using the following definitions:

$$\frac{u^{(j)}}{\bar{q}} = \xi^{(j)}$$

$$\frac{v^{(j)}}{r\bar{p}\bar{q}} = \eta^{(j)}$$

$$rw^{(j)} = \zeta^{(j)}$$

$$\frac{\rho^{(j)}}{\bar{p}} = \chi^{(j)}$$

(II-35)

$$\frac{P^{(j)}}{r\bar{p}} = \pi^{(j)}$$

$$S^{(j)} = S^{(j)}$$

where $j = 1, 2, 3$ and u, v, w , are respectively the axial, radial and tangential components of the velocity together with Equation (II-34) and the expressions derived in Appendix D, the first, second and third order equations can be written in the following general form:

Continuity:

$$\begin{aligned} & \omega^{(j)}(\chi^{(j)})_y + \bar{q}^2(\chi^{(j)})_\phi + \bar{q}^2(\xi^{(j)})_\phi + 2\bar{p}\bar{q}(\psi\eta^{(j)})_\psi + \frac{\bar{p}\bar{q}}{2\psi}(\xi^{(j)})_\phi \\ & = E^{(j)} - \sum_{\substack{i+k=j \\ i,k \neq 0}} \omega^{(i)}(\chi^{(k)})_y \end{aligned} \quad (II-36)$$

Axial Momentum:

$$\begin{aligned} & \omega^{(j)}(\xi^{(j)})_y + (\bar{q}^2\xi^{(j)})_\phi + (\pi^{(j)})_\phi - \frac{1}{2}\frac{d\bar{q}^2}{d\phi}S^{(j)} = (A^{(j)})_\phi - \sum_{\substack{i+k=j \\ i,k \neq 0}} \omega^{(i)}(\xi^{(k)})_y \end{aligned} \quad (II-37)$$

Radial Momentum:

$$\begin{aligned} & \omega^{(j)}(\eta^{(j)})_y + \bar{q}^2(\eta^{(j)})_\phi + (\pi^{(j)})_\psi = (B^{(j)})_\psi - \sum_{\substack{i+k=j \\ i,k \neq 0}} \omega^{(i)}(\eta^{(k)})_y \end{aligned} \quad (II-38)$$

Tangential Momentum:

$$\omega^{(i)}(S^{(i)})_y + \bar{g}^2(S^{(i)})_\varphi + (\pi^{(i)})_\theta = (C^{(i)})_\theta - \sum_{\substack{i+k=j \\ i, k \neq 0}} \omega^{(i)}(S^{(k)})_y \quad (\text{II-39})$$

Entropy Equation:

$$\omega^{(i)}(S^{(i)})_y + \bar{g}^2(S^{(i)})_\varphi = D^{(i)} - \sum_{\substack{i+k=j \\ i, k \neq 0}} \omega^{(i)}(S^{(k)})_y \quad (\text{II-40})$$

Equation of State:

$$\bar{C}^2 S^{(i)} - \pi^{(i)} + \bar{C}^2 \kappa^{(i)} = G^{(i)} \quad (\text{II-41})$$

where the subscripts y , φ , ψ and θ indicate partial derivative with respect to the corresponding coordinate. The exact form of the inhomogeneous parts of the above set of equations will be given now. In first order analysis we get:

$$E^{(1)} = (A^{(1)})_\varphi = (B^{(1)})_\psi = (C^{(1)})_\theta = D^{(1)} = G^{(1)} = 0 \quad (\text{II-42})$$

The inhomogeneous parts of the second order equations are:

$$E^{(2)} = -\bar{g}^2(\kappa^{(1)}\xi^{(1)})_\varphi - 2\bar{f}\bar{g}(\psi\kappa^{(1)}\eta^{(1)})_\psi - \frac{\bar{f}\bar{g}}{2\psi}(\kappa^{(1)}\xi^{(1)})_\theta \quad (\text{II-43a})$$

$$\begin{aligned} (A^{(2)})_\varphi = & \frac{1}{4} \frac{d\bar{g}^2}{d\varphi} \left(\frac{\bar{f}}{\bar{C}^4} \pi^{(1)2} - \kappa^{(1)2} \right) - \frac{1}{2} (\bar{g}^2 \xi^{(1)2})_\varphi - \psi (\bar{f}\bar{g} \eta^{(1)2})_\varphi \\ & - \frac{1}{4\psi} (\bar{f}\bar{g} \xi^{(1)2})_\psi - \frac{1}{2} \bar{f}\bar{g} \left(\frac{\xi^{(1)}}{\psi} ((\xi^{(1)})_\theta - (\xi^{(1)})_\varphi) - 4\psi \eta^{(1)} ((\eta^{(1)})_\psi \right. \\ & \left. - (\xi^{(1)})_\psi) \right) + \kappa^{(1)} ((\pi^{(1)})_\varphi - \frac{1}{2} S^{(1)} \frac{d\bar{g}^2}{d\varphi}) \end{aligned} \quad (\text{II-43b})$$

$$(B^{(2)})_\psi = -\bar{f}\bar{g}(\psi\eta^{(1)2})_\psi - \frac{1}{4} \left(\frac{\xi^{(1)}}{\psi} \right)_\psi - \bar{g}^2 \xi^{(1)}(\eta^{(1)})_\varphi + \kappa^{(1)}(\pi^{(1)})_\psi \quad (\text{II-43c})$$

$$(C^{(2)})_{\theta} = -\bar{f}\bar{g}\psi(\eta^{(1)2})_{\theta} - \frac{1}{4\psi}\bar{f}\bar{g}(\zeta^{(1)2})_{\theta} - \bar{g}^2\zeta^{(1)}(\zeta^{(1)})_{\varphi} + \kappa^{(1)}(\pi^{(1)})_{\theta} \quad (\text{II-43d})$$

$$D^{(2)} = -\bar{g}^2\zeta^{(1)}(s^{(1)})_{\varphi} - 2\bar{f}\bar{g}\psi\eta^{(1)}(s^{(1)})_{\psi} - \frac{\bar{f}\bar{g}}{2\psi}\zeta^{(1)}(s^{(1)})_{\theta} \quad (\text{II-43e})$$

$$G^{(2)} = -\frac{1}{2}\bar{c}^2\left(\frac{f}{\bar{c}^4}\pi^{(1)2} - \kappa^{(1)2}\right) \quad (\text{II-43f})$$

The inhomogeneous parts of the third order equations are:

$$\begin{aligned} E^{(3)} = & -\bar{g}^2(\kappa^{(1)}\zeta^{(2)} + \kappa^{(2)}\zeta^{(1)})_{\varphi} - 2\bar{f}\bar{g}(\psi(\kappa^{(1)}\eta^{(2)} + \kappa^{(2)}\eta^{(1)}))_{\psi} \\ & - \frac{\bar{f}\bar{g}}{2\psi}(\kappa^{(1)}\zeta^{(2)} + \kappa^{(2)}\zeta^{(1)})_{\theta} \end{aligned} \quad (\text{II-44a})$$

$$\begin{aligned} (A^{(3)})_{\varphi} = & \left(\frac{f}{\bar{c}^4}\pi^{(1)}\pi^{(2)} - \frac{1}{3}\frac{f^2}{\bar{c}^6}\pi^{(1)3} - \kappa^{(1)}\kappa^{(2)} + \frac{1}{3}\kappa^{(1)3}\right)\frac{1}{2}\frac{d\bar{g}^{-2}}{d\varphi} \\ & + \kappa^{(1)}\left((\pi^{(2)})_{\varphi} - (s^{(2)} + \frac{1}{2}\left(\frac{f}{\bar{c}^4}\pi^{(1)2} - \kappa^{(1)2}\right)\frac{1}{2}\frac{d\bar{g}^{-2}}{d\varphi}) + (\kappa^{(2)} - \kappa^{(1)2})(\pi^{(1)})_{\varphi} \right. \\ & \left. - \frac{1}{2}s^{(1)}\frac{d\bar{g}^{-2}}{d\varphi}\right) - L_{\varphi} \end{aligned} \quad (\text{II-44b})$$

$$(B^{(3)})_{\psi} = \kappa^{(1)}(\pi^{(2)})_{\psi} + (\kappa^{(2)} - \kappa^{(1)2})(\pi^{(1)})_{\psi} - L_{\psi} \quad (\text{II-44c})$$

$$(C^{(3)})_{\theta} = \kappa^{(1)}(\pi^{(2)})_{\theta} + (\kappa^{(2)} - \kappa^{(1)2})(\pi^{(1)})_{\theta} - L_{\theta} \quad (\text{II-44d})$$

$$\begin{aligned} D^{(3)} = & -\bar{g}^2(\zeta^{(1)}(s^{(2)})_{\varphi} + \zeta^{(2)}(s^{(1)})_{\varphi}) - 2\bar{f}\bar{g}\psi(\eta^{(1)}(s^{(2)})_{\psi} + \eta^{(2)}(s^{(1)})_{\psi}) \\ & - \frac{\bar{f}\bar{g}}{2\psi}(\zeta^{(1)}(s^{(2)})_{\theta} + \zeta^{(2)}(s^{(1)})_{\theta}) \end{aligned} \quad (\text{II-44e})$$

$$G^{(3)} = \frac{1}{3} \frac{\delta}{\bar{c}^4} \pi^{(1)3} - \frac{\delta}{\bar{c}^2} \pi^{(1)} \pi^{(2)} - \frac{1}{3} \bar{c}^2 \chi^{(1)3} + \bar{c}^2 \chi^{(1)} \chi^{(2)} \quad (\text{II-44f})$$

where

$$L = \bar{\rho}^2 \xi^{(1)} \xi^{(2)} + 2 \bar{\rho} \bar{\rho} \eta^{(1)} \eta^{(2)} + \frac{\bar{\rho} \bar{\rho}}{2 \psi} \zeta^{(1)} \zeta^{(2)} \quad (\text{II-44g})$$

The variables appearing in Equations (II-36) through (II-41) represent physically meaningful quantities and consequently must be real quantities. However, since in this case the oscillations are periodic, it will be more convenient to solve these equations by use of complex variables. The final solutions will then be available in complex form. The calculation of the real part of these solutions is one of the objectives of this work. The use of complex variables leads to a considerable simplification of the analysis. This fact, as well as some of the other techniques used in the solution of the equations encountered in this work, are illustrated in the example solved in Appendix A.

The solutions of Equations (II-36) through (II-41), for $j = 1, 2, 3$, represent the wave motion (up to third order) inside the nozzle. These waves, which are generated inside the combustion chamber, are periodic in time and consequently their solutions can be represented by a collection of terms which are proportional to $\exp(ikmy)$ where m and k are integers. In this analysis, the first order solution will be represented by one term proportional to $\exp(imy)$, where m is an arbitrary integer. The form of the second and third order solutions is determined from the form of the inhomogeneous parts of the partial differential equations[#] controlling their behavior. In this case it will be shown that the second and third order solutions can be represented by a summation of terms each of which is proportional to $\exp(ikmy)$ where $k = 0, 1, 2$ in the second order analysis and $k = 1, 3$ in the third order analysis.

[#] The particular solutions that result from the presence of terms proportional to $\omega^{(j)}$ in the inhomogeneous parts of Equations (II-36) through (II-40) will be obtained in a later section

For example we can write for $\xi^{(j)}$:

$$\xi^{(j)}(\phi, \psi, \theta, y) = \sum_k e^{ikmy} \xi_{(km)}^{(j)}(\phi, \psi, \theta)$$

Substituting such series expansions into Equations (II-36) through (II-41), separating them according to the powers of the exponentials and cancelling the exponentials results in the following system of partial differential equations: #

$$ikm\omega^{(0)} \kappa_{(km)}^{(j)} + \bar{q}^2(\kappa_{(km)}^{(j)})_{\phi} + \bar{q}^2(\xi_{(km)}^{(j)})_{\phi} + 2\bar{p}\bar{q}(\psi\eta_{(km)}^{(j)})_{\psi} + \frac{\bar{p}\bar{q}}{2\psi}(\xi_{(km)}^{(j)})_{\theta} = E_{(km)}^{(j)} \quad (\text{II-45})$$

$$ikm\omega^{(0)} \xi_{(km)}^{(j)} + (\bar{q}^2 \xi_{(km)}^{(j)})_{\phi} + (\pi_{(km)}^{(j)})_{\phi} - \frac{1}{2} S_{(km)}^{(j)} \frac{d\bar{q}^2}{d\phi} = (A_{(km)}^{(j)})_{\phi} \quad (\text{II-46})$$

$$ikm\omega^{(0)} \eta_{(km)}^{(j)} + \bar{q}^2(\eta_{(km)}^{(j)})_{\phi} + (\pi_{(km)}^{(j)})_{\psi} = (B_{(km)}^{(j)})_{\psi} \quad (\text{II-47})$$

$$ikm\omega^{(0)} \xi_{(km)}^{(j)} + \bar{q}^2(\xi_{(km)}^{(j)})_{\phi} + (\pi_{(km)}^{(j)})_{\theta} = (C_{(km)}^{(j)})_{\theta} \quad (\text{II-48})$$

$$ikm\omega^{(0)} S_{(km)}^{(j)} + \bar{q}^2(S_{(km)}^{(j)})_{\phi} = D_{(km)}^{(j)} \quad (\text{II-49})$$

$$\bar{c}^2(S_{(km)}^{(j)} + \kappa_{(km)}^{(j)}) - \pi_{(km)}^{(j)} = G_{(km)}^{(j)} \quad (\text{II-50})$$

The subscript (km) was included in the above equations to indicate the time dependence of the particular terms. For convenience it will be omitted, however, from the following analysis. The exact

The particular solutions that result from the presence of terms proportional to $\omega^{(0)}$ in the inhomogeneous parts of Equations (II-36) through (II-40) will be obtained in a later section.

form of the inhomogeneous parts of Equations (II-45) through (II-50) for $j = 1, 2, 3$ will be given in the sections dealing with the detailed solution of these equations.

At this point it will be convenient to define the following "Quasi-Potentials":

$$F^{(j)}(\phi, \psi, \theta) = \int_{\phi_r}^{\phi} \xi^{(j)}(\phi', \psi, \theta) d\phi' + F_1^{(j)}(\psi, \theta) \quad (\text{II-51})$$

$$\tilde{\eta}^{(j)}(\phi, \psi, \theta) = \int_{\psi_r}^{\psi} \eta^{(j)}(\phi, \psi', \theta) d\psi' + F_2^{(j)}(\phi, \theta) \quad (\text{II-52})$$

$$\tilde{\zeta}^{(j)}(\phi, \psi, \theta) = \int_{\theta_r}^{\theta} \zeta^{(j)}(\phi, \psi, \theta') d\theta' + F_3^{(j)}(\phi, \psi) \quad (\text{II-53})$$

where $F_1^{(j)}$, $F_2^{(j)}$ and $F_3^{(j)}$ are arbitrary functions of their arguments. Integrating Equations (II-46), (II-47) and (II-48) with respect to ϕ , ψ and θ respectively yields:

$$ikm\omega^{(j)} F^{(j)} + \bar{q}^2 (F^{(j)})_{\phi} + \pi^{(j)} = \int_0^{\phi} \frac{1}{2} S^{(j)} \frac{d\bar{q}^2}{d\phi} d\phi' + A^{(j)} \quad (\text{II-46a})$$

$$ikm\omega^{(j)} \tilde{\eta}^{(j)} + \bar{q}^2 (\tilde{\eta}^{(j)})_{\psi} + \pi^{(j)} = B^{(j)} \quad (\text{II-47a})$$

$$ikm\omega^{(j)} \tilde{\zeta}^{(j)} + \bar{q}^2 (\tilde{\zeta}^{(j)})_{\theta} + \pi^{(j)} = C^{(j)} \quad (\text{II-48a})$$

The arbitrary functions of the integration are missing from the above equations since they are assumed to be "absorbed" by the arbitrary functions included in the definitions of $F^{(j)}$, $\tilde{\eta}^{(j)}$ and $\tilde{\xi}^{(j)}$.

Integrating Equations (II-47a), (II-48a) and (II-49) with respect to φ gives the following results:

$$\tilde{\eta}^{(j)}(\varphi, \psi, \theta) = f_0^{(km)} \left\{ \int_0^\varphi \frac{B^{(j)} - \pi^{(j)}}{\bar{q}^2 f_0^{(km)}} d\varphi' + \tilde{\eta}^{(j)}(0, \psi, \theta) \right\} \quad (\text{II-54})$$

$$\tilde{\xi}^{(j)}(\varphi, \psi, \theta) = f_0^{(km)} \left\{ \int_0^\varphi \frac{C^{(j)} - \pi^{(j)}}{\bar{q}^2 f_0^{(km)}} d\varphi' + \tilde{\xi}^{(j)}(0, \psi, \theta) \right\} \quad (\text{II-55})$$

$$S^{(j)}(\varphi, \psi, \theta) = f_0^{(km)} \left\{ \int_0^\varphi \frac{D^{(j)}}{\bar{q}^2 f_0^{(km)}} d\varphi' + S^{(j)}(0, \psi, \theta) \right\} \quad (\text{II-56})$$

where

$$f_0^{(km)} = e^{-\int_0^\varphi \frac{ik_m \omega^{(j)}}{\bar{q}^2} d\varphi'} \quad (\text{II-57})$$

Using Equations (II-52), (II-53), (II-54), (II-55) and (II-46a) we get the following expressions for the transverse components of velocity:

$$\eta^{(j)} = f_0^{(km)}(\varphi) \frac{\partial}{\partial \psi} \left\{ \int_0^\varphi \frac{B^{(j)} - H^{(j)}}{\bar{q}^2 f_0^{(km)}} \partial \varphi' + \frac{F^{(j)}}{f_0^{(km)}} - F^{(j)}(0, \psi, \theta) \right\} + f_0^{(km)}(\varphi) \eta^{(j)}(0, \psi, \theta) \quad (\text{II-58})$$

$$\zeta^{(j)} = f_0^{(km)}(\varphi) \frac{\partial}{\partial \theta} \left\{ \int_0^\varphi \frac{C^{(j)} - H^{(j)}}{\bar{q}^2 f_0^{(km)}} \partial \varphi' + \frac{F^{(j)}}{f_0^{(km)}} - F^{(j)}(0, \psi, \theta) \right\} + f_0^{(km)}(\varphi) \zeta^{(j)}(0, \psi, \theta) \quad (\text{II-59})$$

where

$$H^{(j)} = \frac{1}{2} \int_0^\varphi \frac{d\bar{q}^2}{d\varphi} S^{(j)} \partial \varphi' + A^{(j)} \quad (\text{II-60})$$

Using Equations (II-50), (II-46a), (II-51), (II-58), (II-59) and (II-60) together with the solutions of the steady state equations to evaluate the terms appearing in Equation (II-45) results in the derivation of the following inhomogeneous, linear, partial differential equation which controls the behavior of $F^{(j)}$:

$$\begin{aligned} & \bar{q}^2 (\bar{c}^2 - \bar{q}^2) F_{\varphi\varphi}^{(j)} - \bar{q}^2 (2ikm\omega^{(0)} + \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi}) F_{\varphi}^{(j)} + (k^2 m^2 \omega^{(0)2} \\ & - \frac{\bar{q}^2}{\bar{c}^2} \frac{r-1}{2} \frac{d\bar{q}^2}{d\varphi} ikm\omega^{(0)}) F^{(j)} + 2\bar{p}\bar{q}\bar{c}^2 \left(\frac{\partial}{\partial \psi} (\psi \frac{\partial}{\partial \psi}) + \frac{1}{4\psi} \frac{\partial^2}{\partial \theta^2} \right) F^{(j)} = \bar{c}^2 (D^{(j)} + E^{(j)}) \\ & - \bar{q}^2 \left(\frac{G^{(j)}}{\bar{c}^2} + \frac{H^{(j)}}{\bar{c}^2} \right)_{\varphi} - ikm\omega^{(0)} (G^{(j)} + H^{(j)}) - 2\bar{p}\bar{q}\bar{c}^2 \left(\frac{\partial}{\partial \psi} (\psi \frac{\partial}{\partial \psi}) (f_0^{(km)} \int_0^\varphi \frac{B^{(j)} - H^{(j)}}{\bar{q}^2 f_0^{(km)}} \partial \varphi') \right) \\ & + \frac{1}{4\psi} \frac{\partial^2}{\partial \theta^2} (f_0^{(km)} \int_0^\varphi \frac{C^{(j)} - H^{(j)}}{\bar{q}^2 f_0^{(km)}} \partial \varphi') + 2\bar{p}\bar{q}\bar{c}^2 \left(\frac{\partial}{\partial \psi} (\psi \frac{\partial}{\partial \psi}) + \frac{1}{4\psi} \frac{\partial^2}{\partial \theta^2} \right) f_0^{(km)} F^{(j)}(0, \psi, \theta) \\ & - 2\bar{p}\bar{q}\bar{c}^2 f_0^{(km)} \left(\frac{\partial}{\partial \psi} (\psi \frac{\partial}{\partial \psi}) \tilde{\eta}^{(j)}(0, \psi, \theta) + \frac{1}{4\psi} \frac{\partial^2}{\partial \theta^2} \tilde{\zeta}^{(j)}(0, \psi, \theta) \right) = I^{(j)} \end{aligned} \quad (\text{II-61})$$

Substituting the appropriate expressions into its inhomogeneous part, Equation (II-61) will be used to obtain the solution of the first second and third order equations.

Before proceeding with the solution of Equation (II-61), it will be interesting (and important) to derive the expressions describing the vorticity of the flow.

Vorticity of the Flow

Using the power series for \vec{q} we can write

$$\nabla \times \vec{q} = \sum_{j=1}^{\infty} (\nabla \times \vec{q}^{(j)}) \epsilon^{(j)} \quad (\text{II-62})$$

since $\nabla \times \vec{q} = 0$. Each of the coefficients appearing in Equation (II-62) can be written in the following form:

$$\nabla \times \vec{q}^{(j)} = \begin{vmatrix} \frac{1}{\bar{r}} \underline{e}_{\phi} & \frac{1}{r\bar{r}\bar{g}} \underline{e}_{\psi} & r \underline{e}_{\theta} \\ \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial \theta} \\ \xi^{(j)} & \eta^{(j)} & \zeta^{(j)} \end{vmatrix}$$

$$= \bar{r}\bar{g} \left[\frac{\partial}{\partial \psi} (\zeta^{(j)}) - \frac{\partial}{\partial \theta} (\eta^{(j)}) \right] \underline{e}_{\phi}$$

$$+ \frac{1}{r\bar{r}} \left[\frac{\partial}{\partial \theta} (\xi^{(j)}) - \frac{\partial}{\partial \phi} (\zeta^{(j)}) \right] \underline{e}_{\psi}$$

$$+ r\bar{r}\bar{g} \left[\frac{\partial}{\partial \phi} (\eta^{(j)}) - \frac{\partial}{\partial \psi} (\xi^{(j)}) \right] \underline{e}_{\theta} \quad (\text{II-63})$$

In order to examine under what conditions the flow is irrotational the solutions for the expressions appearing inside the square brackets in Equation (II-63) will be obtained. Differentiating Equations (II-47) with respect to θ and Equation (II-48) with respect to ψ and subtracting them from one another gives:

$$(\xi_{\psi}^{(j)} - \eta_{\theta}^{(j)})_{\varphi} + \frac{ik_m \omega^{(j)}}{\bar{q}^2} (\xi_{\psi}^{(j)} - \eta_{\theta}^{(j)}) = \frac{1}{\bar{q}^2} (C_{\theta\psi}^{(j)} - B_{\psi\theta}^{(j)}) \quad (\text{II-64})$$

The solution of Equation (II-64) is:

$$\xi_{\psi}^{(j)} - \eta_{\theta}^{(j)} = f_0^{(km)}(\varphi) \left\{ \int_0^{\varphi} \frac{C_{\theta\psi}^{(j)} - B_{\psi\theta}^{(j)}}{\bar{q}^2 f_0^{(km)}} d\varphi' + \xi_{\psi}^{(j)}(0, \psi, \theta) - \eta_{\theta}^{(j)}(0, \psi, \theta) \right\} \quad (\text{II-65})$$

Performing similar manipulations with Equations (II-46), (II-47) and (II-48) gives the following solutions for the other components of the vorticity vector:

$$\xi_{\theta}^{(j)} - \xi_{\varphi}^{(j)} = \tilde{f}_0^{(km)}(\varphi) \left\{ \int_0^{\varphi} \frac{A_{\theta\varphi}^{(j)} - C_{\theta\varphi}^{(j)}}{\bar{q}^2 \tilde{f}_0^{(km)}} d\varphi' + \frac{1}{2} \int_0^{\varphi} \frac{S_{\theta}^{(j)} \frac{d\bar{q}^2}{d\varphi'}}{\bar{q}^2 \tilde{f}_0^{(km)}} d\varphi' + (\xi_{\theta}^{(j)}(0, \psi, \theta) - \xi_{\varphi}^{(j)}(0, \psi, \theta)) \right\} \quad (\text{II-66})$$

and

$$\eta_{\varphi}^{(j)} - \xi_{\psi}^{(j)} = \tilde{f}_0^{(km)} \left\{ \int_0^{\varphi} \frac{B_{\psi\varphi}^{(j)} - A_{\psi\varphi}^{(j)}}{\bar{q}^2 \tilde{f}_0^{(km)}} d\varphi' - \frac{1}{2} \int_0^{\varphi} \frac{S_{\psi}^{(j)} \frac{d\bar{q}^2}{d\varphi'}}{\bar{q}^2 \tilde{f}_0^{(km)}} d\varphi' + (\eta_{\varphi}^{(j)}(0, \psi, \theta) - \xi_{\psi}^{(j)}(0, \psi, \theta)) \right\} \quad (\text{II-67})$$

where

$$\tilde{f}_0^{(km)}(\varphi) = \frac{\bar{q}_0^2(\varphi)}{\bar{q}_0^2(\varphi)} e^{-\int_0^\varphi \frac{ikm\omega^{(k)}}{\bar{q}_0^2(\varphi)} d\varphi} = \frac{\bar{q}_0^2(\varphi)}{\bar{q}_0^2(\varphi)} f_0^{(km)}(\varphi) \quad (\text{II-68})$$

Examination of Equations (II-62), (II-64) and (II-65) shows that the following are necessary and sufficient conditions for the disappearance of the vorticity:

$$C_{\theta\psi}^{(j)} = B_{\psi\theta}^{(j)} \quad \text{for all } \varphi \quad (\text{II-69a})$$

$$\xi_{\varphi}^{(j)}(0, \psi, \theta) - \eta_{\theta}^{(j)}(0, \psi, \theta) = 0 \quad (\text{II-69b})$$

$$A_{\varphi\theta}^{(j)} = C_{\theta\varphi}^{(j)} \quad \text{for all } \varphi \quad (\text{II-70a})$$

$$\xi_{\theta}^{(j)}(0, \psi, \theta) - \xi_{\varphi}^{(j)}(0, \psi, \theta) = 0 \quad (\text{II-70b})$$

$$S_{\theta}^{(j)} = 0 \quad \text{for all } \varphi \quad (\text{II-70c})$$

$$B_{\psi\varphi}^{(j)} = A_{\varphi\psi}^{(j)} \quad \text{for all } \varphi \quad (\text{II-71a})$$

$$\eta_{\varphi}^{(j)}(0, \psi, \theta) - \xi_{\psi}^{(j)}(0, \psi, \theta) = 0 \quad (\text{II-71b})$$

$$S_{\psi}^{(j)} = 0 \quad \text{for all } \varphi \quad (\text{II-71c})$$

For reasons that will become clear shortly the location $\varphi = 0$ is taken to be at the nozzle throat. Consequently the above conditions imply that in the case of irrotational flow the vorticity must be zero there. In reality, the vorticity is generated inside the

combustion chamber by the combustion process and then transported by the flow into the nozzle. Consequently zero vorticity at the nozzle throat implies that no vorticity is being generated by the combustion process. Equations (II-70c) and (II-71c) are in agreement with the well known fluid mechanical theorem that states that potential flow must be homoentropic. These equations can be replaced by the more general requirement that the entropy perturbations be identically zero. It is interesting to note, however, that the coupling between the entropy terms and the vorticity (or momentum equation) is through $\frac{d\bar{q}}{d\phi}^2$. Consequently when the mean steady-state flow is constant a situation can exist in which the flow will be irrotational in spite of the fact that the entropy perturbations are nonzero. This situation will be encountered in the analysis of the combustion chamber flow where all the combustion is assumed to be concentrated at the injector face and consequently the steady-state flow inside the combustion chamber is assumed to be constant. From Equations (II-69a) and (II-69b) it can be seen that the disappearance of the axial component of vorticity is completely independent of the nature of the entropy of the flow. This phenomenon can be attributed to the choice of the perturbation scheme where the basic, steady-state unperturbed flow is one dimensional.

In order to simplify the solution of Equation (II-61) it will be necessary to assume that the axial component of vorticity is zero (see Equations (II-69a) and (II-69b) for the conditions implied by this assumption). Crocco⁶, in his analysis of the unsteady flow in the combustion chamber, has shown that the presence of a combustion process results in the appearance of only the transverse components of vorticity. It is thus expected that in reality the above assumption will be approximately satisfied.

Using the definitions of $\tilde{g}^{(j)}$ and $\tilde{\eta}^{(j)}$ Equation (II-69b) gives:

$$\tilde{g}_{\theta\psi}^{(j)}(0, \psi, \theta) = \tilde{g}_{\psi\theta}^{(j)}(0, \psi, \theta) = \tilde{\eta}_{\psi\theta}^{(j)}(0, \psi, \theta)$$

which implies

$$\tilde{\eta}_{\theta}^{(j)}(0, \psi, \theta) = \tilde{\zeta}_{\theta}^{(j)}(0, \psi, \theta) \quad (\text{II-72a})^{\#}$$

or

$$\tilde{\eta}_{\theta\theta}^{(j)}(0, \psi, \theta) = \tilde{\zeta}_{\theta\theta}^{(j)}(0, \psi, \theta) \quad (\text{II-72b})$$

By use of Equation (II-69a) and performing the same manipulations as above it can be shown that when the axial component of vorticity is identically zero the following additional relation must also hold:

$$B_{\theta\theta}^{(j)} = C_{\theta\theta}^{(j)} \quad (\text{II-73})^{\#\#}$$

Substituting Equations (II-72b) and (II-73) into the inhomogeneous part of Equation (II-61) gives:

$$\begin{aligned} I^{(j)} = & \bar{C}^2 \left(D^{(j)} + E^{(j)} - \bar{q}^2 \left(\frac{1}{\bar{c}^2} G^{(j)} + \frac{1}{\bar{c}^2} H^{(j)} \right)_{\varphi} \right) \\ & - i k_m \omega^{(j)} (G^{(j)} + H^{(j)}) - 2 \bar{p} \bar{q} f_0^{(km)} \left\{ \left(\frac{\partial}{\partial \psi} (\psi \frac{\partial}{\partial \psi}) \right. \right. \\ & \left. \left. + \frac{1}{4\psi} \frac{\partial^2}{\partial \theta^2} \right) \left(\int_0^{\varphi} \frac{B_{\theta\theta}^{(j)} - H_{\theta\theta}^{(j)}}{\bar{q}^2 f_0^{(km)}} \partial \varphi' + \tilde{\eta}_{\theta}^{(j)}(0, \psi, \theta) - \tilde{F}_{\theta}^{(j)}(0, \psi, \theta) \right) \right\} \end{aligned} \quad (\text{II-74})$$

which represents a considerably simpler form of $I^{(j)}$.

[#] In deriving this relation, the arbitrary function of the integration was taken to be identically zero. This relation will be shown rigorously in the sections dealing with the solution of the specific equations (for $j = 1, 2, 3$).

^{\#\#} The above footnote applies here also. In addition, it is possible to have a situation in which Equation (II-73) holds but Equation (II-69a) does not hold.

First Order Solution

Assuming that the first order solution is proportional to $\exp(imy)$ we get from Equation (II-56):

$$S'' = S''(0, \psi, \theta) e^{-\int_0^\varphi i m \omega'' \frac{d\varphi'}{\bar{g}^2}} = S''(0, \psi, \theta) f_0^{(m)}(\varphi) \quad (\text{II-75})$$

since $D^{(1)} = 0$.

Substituting Equation (II-75) into Equation (II-60) and using Equation (II-42) gives

$$\begin{aligned} H'' &= \int_0^\varphi \frac{1}{2} \frac{d\bar{g}^2}{d\varphi'} S'' d\varphi' = S''(0, \psi, \theta) \int_0^\varphi \frac{1}{2} \frac{d\bar{g}^2}{d\varphi'} f_0^{(m)}(\varphi') d\varphi' \\ &= S''(0, \psi, \theta) f_1^{(m)}(\varphi) \end{aligned} \quad (\text{II-76})$$

where

$$f_1^{(km)}(\varphi) = \frac{1}{2} \int_0^\varphi \frac{d\bar{g}^2}{d\varphi'} f_0^{(km)}(\varphi') d\varphi' \quad (\text{II-77})$$

In order to evaluate $I^{(1)}$, the following expression will be necessary:

$$\int_0^\varphi \frac{H''}{\bar{g}^2 f_0^{(m)}} d\varphi' = S''(0, \psi, \theta) \int_0^\varphi \frac{f_1^{(m)}}{\bar{g}^2 f_0^{(m)}} d\varphi' = S''(0, \psi, \theta) \frac{f_2^{(m)}(\varphi)}{f_0^{(m)}(\varphi)} \quad (\text{II-78})$$

where

$$f_2^{(km)}(\varphi) = f_0^{(km)}(\varphi) \int_0^\varphi \frac{f_1^{(km)}(\varphi')}{\bar{g}^2 f_0^{(km)}(\varphi')} d\varphi' \quad (\text{II-79})$$

Substituting the relations derived in this section into Equation (II-74) and using the results of Equation (II-42) gives:

$$I'' = \bar{C}^2 S''_{(0,\psi,\theta)} \left(-\bar{q}^2 \left(\frac{f_1^{(m)}}{\bar{C}^2} \right)_\varphi - im\omega^{(m)} \left(\frac{f_1^{(m)}}{\bar{C}^2} \right) \right) + 2\bar{F}\bar{q}\bar{C}^2 \left\{ \left(\frac{\partial}{\partial\psi} (\psi \frac{\partial}{\partial\psi}) + \frac{1}{4\psi} \frac{\partial^2}{\partial\theta^2} \right) \left(S''_{(0,\psi,\theta)} f_2^{(m)} - f_0^{(m)} (\tilde{\eta}''_{(0,\psi,\theta)} - F''_{(0,\psi,\theta)}) \right) \right\} \quad (II-80)$$

Defining the following function

$$f_3^{(km)}(\varphi) = \frac{f_1^{(km)}(\varphi)}{\bar{C}^2 f_0^{(km)}(\varphi)} \quad (II-81)$$

and using it to evaluate the first two terms in Equation (II-80) gives:

$$\begin{aligned} -\bar{q}^2 (f_0^{(m)} f_3^{(m)})_\varphi - im\omega^{(m)} f_0^{(m)} f_3^{(m)} &= -f_3^{(m)} (\bar{q}^2 f_0^{(m)} + im\omega^{(m)} f_0^{(m)}) \\ -\bar{q}^2 f_0^{(m)} (f_3^{(m)})_\varphi &= -\bar{q}^2 f_0^{(m)} (f_3^{(m)})_\varphi \end{aligned} \quad (II-82)$$

since

$$\bar{q}^2 (f_0^{(m)})_\varphi + im\omega^{(m)} f_0^{(m)} = 0 \quad (II-83)$$

In order to solve Equation (II-61) (for $j = 1$) by the method of separation of variables, it will be assumed that the first order variables can be written in the following form:

$$F'' = \underline{F}''(\varphi) K''(\psi, \theta, y)$$

$$\underline{S}'' = U''(\varphi) K''(\psi, \theta, y)$$

$$\tilde{\eta}'' = V''(\varphi) K''(\psi, \theta, y)$$

(II-84)[#]

[#] Equation (II-84) continued on next page .

$$\tilde{S}^{(1)} = W^{(1)}(\varphi) K^{(1)}(\psi, \theta, y)$$

$$\Pi^{(1)} = P^{(1)}(\varphi) K^{(1)}(\psi, \theta, y)$$

$$K^{(1)} = R^{(1)}(\varphi) K^{(1)}(\psi, \theta, y)$$

$$S^{(1)} = S^{(1)}(\varphi) K^{(1)}(\psi, \theta, y)$$

where

$$K^{(1)}(\psi, \theta, y) = \Theta^{(1)}(\theta) \bar{\Psi}^{(1)}(\psi) e^{imy}$$

(II-85)

$$\Theta^{(1)}(\theta) = \frac{\cos \nu \theta}{\sin \nu \theta}$$

for the case of standing-wave motion

and

$$\Theta^{(1)}(\theta) = e^{\pm i\nu \theta}$$

for the case of travelling-wave motion

(II-86)

and

$$\bar{\Psi}^{(1)}(\psi) = J_{\nu}(S_{(\nu,h)} \sqrt{\frac{\psi'}{\psi_w}})$$

(II-87)

is a Bessel function of order ν which satisfies the following ordinary differential equation:

$$\psi \frac{d^2}{d\psi^2} J_{\nu}(S_{(\nu,h)} \sqrt{\frac{\psi'}{\psi_w}}) + \frac{d}{d\psi} J_{\nu}(S_{(\nu,h)} \sqrt{\frac{\psi'}{\psi_w}}) + \left(\frac{S_{(\nu,h)}^2}{4\psi_w} - \frac{\nu^2}{4\psi} \right) J_{\nu}(S_{(\nu,h)} \sqrt{\frac{\psi'}{\psi_w}}) = 0$$

(II-88)

ψ_w is the value of the steady-state stream function evaluated at the nozzle wall and $S_{(\nu,h)}$ is a root of the equation $\frac{d}{dx} J_{\nu}(x) = 0$.

Using Equation (II-84) and the definitions of $\tilde{\eta}^{(1)}$ and $\tilde{\xi}^{(1)}$ one gets:

$$\xi'' = V''(\varphi) \frac{d}{d\theta} \Theta''(\theta) \bar{\Psi}''(\psi) e^{imy} \quad (\text{II-89})$$

$$\eta'' = W''(\varphi) \Theta''(\theta) \frac{d}{d\psi} \Psi''(\psi) e^{imy} \quad (\text{II-90})$$

Note that in the given form the first order solutions are periodic with θ and y and $\eta^{(1)}$ (which is essentially equal to the radial component of velocity) vanishes at the nozzle wall. The functions defined in Equation (II-84) are complex functions of their arguments.

With the available information, we can now proceed with the solution of Equation (II-61) for the case $j = 1$. Substituting Equations (II-80), (II-81), (II-82) and (II-84) into Equation (II-61) and noting that

$$\left(\frac{\partial}{\partial \psi} \left(\psi \frac{\partial}{\partial \psi} \right) + \frac{1}{4\psi} \frac{\partial^2}{\partial \theta^2} \right) K''(\psi, \theta, y) = - \frac{S_{(1)}^2}{4\psi_w} K''(\psi, \theta, y)$$

results, after cancelling $K^{(1)}(\psi, \theta, y)$ on both sides of the equation, in the following linear inhomogeneous differential equation for $\Phi^{(1)}(\varphi)$:

$$\begin{aligned} & \bar{q}^2 (\bar{c}^2 - \bar{q}^2) \frac{d^2}{d\varphi^2} \Phi'' - \bar{q}^2 \left(2im\omega^{(1)} + \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} \right) \frac{d}{d\varphi} \Phi'' + (m^2 \omega^{(1)2} \\ & - \frac{\bar{q}^2}{\bar{c}^2} \frac{\bar{q}-1}{2} \frac{d\bar{q}^2}{d\varphi} im\omega^{(1)} - \frac{S_{(1)}^2}{2\psi_w} \bar{p} \bar{q} \bar{c}^2) \Phi'' = -\bar{c}^2 \sigma^{(1)} \left(\bar{q}^2 f_0^{(1)} \left(\frac{f_3^{(1)}}{f_3^{(1)}} \right) \right)_{\varphi} + \frac{S_{(1)}^2}{2\psi_w} \bar{p} \bar{q} f_2 \\ & + C_1 \frac{S_{(1)}^2}{2\psi_w} \bar{p} \bar{q} \bar{c}^2 f_0^{(1)} \end{aligned} \quad (\text{II-91})$$

where

$$\sigma^{(1)} = S''(\varphi) \quad (\text{II-92a})$$

and

$$C_1 = V''(\varphi) - \Phi''(\varphi) \quad (\text{II-92b})$$

are the φ dependent components of the complex entropy and vorticity evaluated at the nozzle throat.[#] Equation (II-91) is identical to the one derived by Crocco⁵ in his derivation of the linear admittance relation. For convenience Equation (II-91) will be rewritten in the following form:

$$\mathcal{L}^{(m)}(\Phi^{(m)}) = \sigma^{(m)} I_e^{(m)} + C^{(m)} I_v^{(m)} \quad (\text{II-93})$$

where

$\mathcal{L}^{(m)}(\Phi^{(1)})$ represents the left side of Equation (II-91) and

$$I_e^{(m)} = -\bar{C}^2 (\bar{q}^2 f_0^{(m)} (f_3^{(m)})_\varphi + \frac{S_{(y_h)}^2}{2\psi_w} \bar{p} \bar{q} f_2) \quad (\text{II-94a})$$

$$I_v^{(m)} = \frac{S_{(y_h)}^2}{2\psi_w} \bar{p} \bar{q} \bar{C}^2 f_0^{(m)} \quad (\text{II-94b})$$

The derivation of the linear admittance relation as well as a discussion of the solution of Equation (II-93) will be given in other sections of this chapter. For completeness sake the expressions for $P_{(\varphi)}^{(1)}$, $V_{(\varphi)}^{(1)}$, $U_{(\varphi)}^{(1)}$ and $R_{(\varphi)}^{(1)}$ will now be derived. From Equation (II-84) and the definition of $F_{(\varphi)}^{(j)}$ it immediately follows that

$$U_{(\varphi)}^{(1)} = \frac{\partial}{\partial \varphi} \Phi_{(\varphi)}^{(1)} \quad (\text{II-95a})$$

Using Equations (II-84), (II-42), (II-46a), (II-76) and (II-77) it can be shown that

$$P_{(\varphi)}^{(1)} = \sigma^{(1)} f_{1(\varphi)}^{(1)} - \bar{q}^2 \frac{d}{d\varphi} \Phi_{(\varphi)}^{(1)} - i m \omega^{(1)} \Phi_{(\varphi)}^{(1)} \quad (\text{II-95b})$$

[#] See Appendix E for verification of this statement.

Substitution of the available solutions of $S^{(1)}$ and $P^{(1)}$ into Equation (II-50) and using the results presented in Equation (II-42) gives

$$R'' = \frac{1}{\epsilon^2} P'' - S''$$

(II-95c)

Finally using Equations (II-84), and (E-1) from Appendix E, to evaluate $V^{(1)} = W^{(1)}$, gives the following result

$$V'' = \Phi'' - \sigma'' f_2^{(m)} + C_1 f_0^{(m)}$$

(II-95d)

after separation of variables.

Second Order Solutions

It was noted previously (and demonstrated through a solution of a specific example in Appendix A) that the use of complex variables will simplify the solution of the equations derived in this analysis. The variables appearing in these equations can all be related to physically meaningful quantities and consequently should be described by real variables. In the second order analysis, the inhomogeneous parts of the equations contain quantities which are products of the real parts of two first order quantities. Since complex variables were also used in the analysis of the first order equations, their solutions are available in complex form. In this case, the product of two first order quantities, $A^{(1)}$ and $B^{(1)}$, which appears in a second order equation will have to be expressed in the following form:

$$A'' B'' = A_r'' B_r'' = \frac{1}{2} (A'' + A''^*) \frac{1}{2} (B'' + B''^*)$$

$$= \frac{1}{4} (A'' B'' + A''^* B'' + A'' B''^* + A''^* B''^*)$$

(II-96a)

where $*$ indicates the complex conjugate of the designated quantity.

When the second order solution is obtained in complex form the expression given in Equation (II-96a) (which represents a typical expression that may be present in the inhomogeneous part of the second order equations) can be replaced by any one of the following expressions:

$$\begin{aligned} A_r'' B_r'' &= \operatorname{Re} \left\{ \frac{1}{2} A'' (B'' + B''^*) \right\} = \operatorname{Re} \left\{ B'' (A'' + A''^*) \right\} \\ &= \operatorname{Re} \left\{ \frac{1}{2} A''^* (B'' + B''^*) \right\} = \operatorname{Re} \left\{ B''^* (A'' + A''^*) \right\} \end{aligned} \quad (\text{II-96b})$$

A comparison between Equations (II-96a) and (II-96b) shows that when the latter is used, the number of terms appearing in the inhomogeneous part of a second order equation is halved and the calculation of the particular solutions becomes considerably easier. Using the procedures outlined above, and some of the relations derived in first order analysis, Equations (II-43a) through (II-43f) will now be rewritten in complex form:[#]

$$\begin{aligned} A_\varphi^{(2)} &= \frac{1}{2} \Psi^{(2)} \left\{ \left(\frac{1}{4} \frac{d\bar{q}^2}{d\varphi} \left(\frac{\delta-1}{\bar{c}^4} P^{(2)} + S^{(2)} \right) + R^{(2)} P^{(2)} - \frac{1}{2} (\bar{q}^2 U^{(2)})' \right) \Theta^{(2)} e^{2im\varphi} \right. \\ &\quad \left. + \left(\frac{1}{4} \frac{d\bar{q}^2}{d\varphi} \left(\frac{\delta-1}{\bar{c}^4} P^{(2)} P^{(2)*} + S^{(2)} S^{(2)*} \right) + R^{(2)} P^{(2)*} - \frac{1}{2} (\bar{q}^2 U^{(2)} U^{(2)*})' \right) \Theta^{(2)} \Theta^{(2)*} \right\} \\ &\quad - \frac{1}{2} \Psi (\Psi')^2 \left\{ (\bar{P}\bar{q})' (V^{(2)} \Theta^{(2)} e^{2im\varphi} + V^{(2)} V^{(2)*} \Theta^{(2)} \Theta^{(2)*}) + 2\bar{P}\bar{q} (V^{(2)} V^{(2)'} \Theta^{(2)} e^{2im\varphi} \right. \\ &\quad \left. + (V^{(2)} V^{(2)*'} + V^{(2)'} V^{(2)*}) \frac{1}{2} \Theta^{(2)} \Theta^{(2)*}) \right\} - \frac{1}{8\bar{\Psi}} \Psi^{(2)} \left\{ (\bar{P}\bar{q})' (V^{(2)} (\Theta^{(2)})^2 e^{2im\varphi} \right. \\ &\quad \left. + V^{(2)} V^{(2)*} \Theta^{(2)} \Theta^{(2)*'}) + \bar{P}\bar{q} (2V^{(2)} V^{(2)'} (\Theta^{(2)})^2 e^{2im\varphi} + (V^{(2)} V^{(2)*} + V^{(2)'} V^{(2)*'}) \Theta^{(2)} \Theta^{(2)*'}) \right\} \end{aligned}$$

(II-97)

[#] In the expressions that follow, the complete form of the inhomogeneous parts of the second order equations is given. Note that each of these expressions can be expressed as a sum of two terms, one proportional to $e^{2im\varphi}$ and the other proportional to $e^{i0m\varphi} = 1$. The analysis that was performed in previous sections applies to each of these terms separately. The additional solutions of second order equations which result from the presence of terms proportional to ω'' in the inhomogeneous parts of the second order equations will be obtained separately.

Integration of Equation (II-43c)[#] with respect to Ψ gives:

$$\begin{aligned}
 B^{(2)} = & \frac{1}{4} \bar{\Psi}^{''2} \left\{ (R^{''} P^{''} - \bar{q}^2 U^{''} V^{''}) \Theta^{''2} e^{2imy} + (R^{''} P^{''*} \right. \\
 & - \bar{q}^2 U^{''} V^{''*}) \Theta^{''} \Theta^{''*} \left. \right\} - \frac{1}{2\psi} \bar{f} \bar{g} \bar{\Psi}^{''2} \left\{ W^{''2} (\Theta^{''})^2 e^{2imy} + W^{''} W^{''*} \Theta^{''} \Theta^{''*} \right\} \\
 & - \frac{1}{2} \bar{f} \bar{g} \psi (\bar{\Psi}^{''})^2 \left\{ V^{''2} \Theta^{''2} e^{2imy} + V^{''} V^{''*} \Theta^{''} \Theta^{''*} \right\}
 \end{aligned}
 \tag{II-98}$$

Integration of Equation (II-43d) with respect to Θ gives:

$$\begin{aligned}
 C^{(2)} = & \frac{1}{4} \bar{\Psi}^{''2} \left\{ (R^{''} P^{''} - \bar{q}^2 U^{''} V^{''}) \Theta^{''2} e^{2imy} \right. \\
 & + (R^{''} P^{''*} - \bar{q}^2 U^{''} V^{''*}) 2 \int_{\Theta_r}^{\Theta} \Theta^{''} \Theta^{''*'} d\Theta' \left. \right\} \\
 & - \frac{1}{2\psi} \bar{f} \bar{g} \bar{\Psi}^{''2} \left\{ W^{''2} \Theta^{''2} e^{2imy} + W^{''} W^{''*} \Theta^{''} \Theta^{''*} \right\} \\
 & - \frac{1}{2} \bar{f} \bar{g} \psi (\bar{\Psi}^{''})^2 \left\{ V^{''2} \Theta^{''2} e^{2imy} + V^{''} V^{''*} \Theta^{''} \Theta^{''*} \right\}
 \end{aligned}
 \tag{II-99}$$

$$\begin{aligned}
 E^{(2)} = & -\frac{1}{2} \left\{ \bar{q}^2 \bar{\Psi}^{''2} \left((R^{''} U^{''})' \Theta^{''2} e^{2imy} + (R^{''} U^{''*})' \Theta^{''} \Theta^{''*} \right) \right. \\
 & + 2\bar{f} \bar{g} \left(\psi (\bar{\Psi}^{''})^2 - \frac{S_{\psi\psi}}{4\psi} \bar{\Psi}^{''2} + \frac{\psi^2}{4\psi} \bar{\Psi}^{''2} \right) (R^{''} V^{''} \Theta^{''2} e^{2imy} + R^{''} V^{''*} \Theta^{''} \Theta^{''*}) \\
 & \left. + \frac{1}{2\psi} \bar{f} \bar{g} \bar{\Psi}^{''2} \left(R^{''} W^{''} (\Theta^{''} \Theta^{''*})' e^{2imy} + R^{''} W^{''*} (\Theta^{''} \Theta^{''*})' \right) \right\}
 \end{aligned}
 \tag{II-100}$$

[#] The arbitrary functions which appear as a result of the integration of $B_{\psi}^{(2)}$ and $C_{\psi}^{(2)}$ are absorbed by $\tilde{\eta}^{(2)}$ and $\tilde{\xi}^{(2)}$ respectively. The definitions of these two variables contain an arbitrary function.

$$\begin{aligned}
 D^{(2)} = & -\frac{1}{2} \left\{ \bar{q}^2 \Psi^{(2)} (U^{(1)} S^{(1)'} \Theta^{(1)2} e^{2im\gamma} + U^{(1)} S^{(1)*'} \Theta^{(1)} \Theta^{(1)*}) \right. \\
 & + 2\bar{p}\bar{q}\bar{\psi}(\Psi^{(1)})^2 (V^{(1)} S^{(1)} \Theta^{(1)2} e^{2im\gamma} + V^{(1)} S^{(1)*} \Theta^{(1)} \Theta^{(1)*}) \\
 & \left. + \frac{1}{2\bar{\psi}} \bar{p}\bar{q} \Psi^{(1)2} (W^{(1)} S^{(1)} (\Theta^{(1)})^2 e^{2im\gamma} + W^{(1)} S^{(1)} \Theta^{(1)'} \Theta^{(1)*'}) \right\}
 \end{aligned}
 \tag{II-101}$$

$$G^{(2)} = -\frac{1}{4} \bar{C}^2 \Psi^{(1)2} \left\{ \left(\frac{r}{\bar{c}^4} P^{(1)2} - R^{(1)2} \right) \Theta^{(1)2} e^{2im\gamma} + \left(\frac{r}{\bar{c}^4} P^{(1)} P^{(1)*} - R^{(1)} R^{(1)*} \right) \Theta^{(1)} \Theta^{(1)*} \right\}
 \tag{II-102}$$

Subtraction of Equation (II-99) from Equation (II-98) shows that when standing waves are being considered

$$B^{(2)} = C^{(2)} \tag{II-103a}$$

while for travelling waves

$$B^{(2)} - C^{(2)} = \frac{1}{4} \Psi^{(1)2} \left(R^{(1)} P^{(1)*} - \bar{q}^2 U^{(1)} V^{(1)*'} \right) (1 + 2i\nu\theta) \tag{II-103b}$$

where Θ_r was taken to be identically zero. In both cases $B_{\theta\theta}^{(2)} = C_{\theta\theta}^{(2)}$ and consequently $I^{(2)}$, as given by Equation (II-74) can be used in solving for $F^{(2)}$ (provided Equation (II-72b) for $j = 2$ is also satisfied). By differentiation of Equation (II-103a) it can be shown that $B_{\theta\psi}^{(2)} = C_{\theta\psi}^{(2)} = C_{\psi\theta}^{(2)}$. It then follows that in the case of standing waves the axial component of vorticity is identically zero. In the case of travelling waves differentiation of Equation (103-b) gives:

$$B_{\psi\theta}^{(2)} - C_{\psi\theta}^{(2)} = \Psi^{(1)} \Psi^{(1)'} (R^{(1)} P^{(1)*} - \bar{q}^2 U^{(1)} V^{(1)*'}) i\nu \tag{II-104}$$

The inhomogeneity of Equations (II-103b) and (II-104) can be attributed to the presence of terms which are independent of time as well as the azimuthal direction in the inhomogeneous parts of the radial and tangential components of the momentum equation which describe the travelling-wave motion. Substitution of Equation (II-104) into Equation (II-65), letting $k = 0$ (since the terms appearing in Equation (II-104) are independent of time) and assuming that relation (II-69b) holds results in the following expression for the second-order axial component of vorticity:

$$\begin{aligned} (\zeta_{\varphi}^{(2)} - \eta_{\theta}^{(2)})_r &= \frac{1}{2} \frac{1}{\bar{q}^2} \int_0^{\varphi} (C_{\varphi\psi}^{(2)} - B_{\varphi\theta}^{(2)} + C_{\varphi\psi}^{(2)*} - B_{\varphi\theta}^{(2)*}) d\varphi' \\ &= \frac{1}{2\bar{q}^2} \Psi'' \Psi''' \int_0^{\varphi} i\nu (\tilde{A}(\varphi') - \tilde{A}(\varphi')^*) d\varphi' \end{aligned} \quad (\text{II-105a})$$

where

$$\tilde{A}(\varphi) = R'' P''^* - \bar{q}^2 U'' V''^* \quad (\text{II-105b})$$

Consequently the second order axial component of vorticity in the case of travelling wave motion, is identically zero if, and only if,

$$A(\varphi) = \tilde{A}(\varphi)^*.$$

To evaluate $A(\varphi)$ some of the relations derived in the first order analysis will be re-examined. Assuming that the first order flow is irrotational (and thus homentropic) and using Equations (II-95c) and (II-95d) gives the following results:

$$\begin{aligned} \bar{C}_R^{(1)} &= P^{(1)} \\ V^{(1)} &= U^{(1)} \end{aligned} \quad (\text{II-106})$$

Using Equations (II-106) and (II-105b) it can be easily shown that indeed $\tilde{A}(\varphi) = \tilde{A}(\varphi)^*$. In conclusion it can be said that when the

first order flow is irrotational and the second order axial component of vorticity at the nozzle throat ($\varphi = 0$) is zero then the second order axial component of vorticity for both the standing and travelling-wave motion is identically zero.

To obtain the solution of the second order flow, an appropriate form of the inhomogeneous part of Equation (II-61), for $j = 2$, must be derived. Substitution of the appropriate parts (i.e., those having the same time dependence) of $A_\varphi^{(2)}$, $B^{(2)}$, $D^{(2)}$, $E^{(2)}$ and $G^{(2)}$ into Equation (II-74) (which gives a more useful form of $I^{(2)}$) and using the latter in Equation (II-61) results in a cumbersome, inhomogeneous partial differential equation for $F^{(2)}$ which can no longer be solved by a straightforward application of the method of Separation of Variables. This difficulty is caused by the presence of products of first order quantities in $I^{(2)}$. The similarities between the equations controlling the behavior of $F^{(1)}$ and $F^{(2)}$ suggest the use of a series solution for $F^{(2)}$. In employing this method the terms appearing in $I^{(2)}$ as well as the second order variables are expanded in a Fourier type series in terms of the eigenfunctions which transform the homogeneous part of the equation from a partial differential equation into an ordinary differential equation. (This method, which is also known as an "Eigenfunction Expansion", is often used in the solution of forced vibration problems⁹.)

Examples of such eigenfunctions are given by the following expressions:

$$\tilde{K}(\psi, \theta) = \cos n\nu\theta \, J_{n\nu} \left(S_{n\nu, q}, \sqrt{\frac{\psi}{\psi_w}} \right)$$

for standing waves

and

$$\tilde{K}(\psi, \theta) = e^{in\nu\theta} J_{n\nu} \left(S_{n\nu, q}, \sqrt{\frac{\psi}{\psi_w}} \right)$$

for travelling waves

(II-107)

where

$$n = 0, 1, 2, \dots$$

$$q = 1, 2, 3, \dots$$

The transverse dependence of the first order solution was expressed by one of these eigenfunctions. These transverse eigenfunctions form a complete, orthogonal set and thus can be used for the expansion of $I^{(2)}$. A more detailed discussion regarding the use of this method can be found in Reference 2 where the same method was used in the investigation of the effect of the transverse velocity components upon linear combustion instability. Proceeding with the solution of the second order equations, the expansion of the part of $B^{(2)}$, see Equation (II-98), which appears in $I^{(2)}$ and is proportional to $\exp(2imy)$ will be demonstrated in detail for the case of standing-wave motion:

$$B_{(2m)}^{(2)} = \left(\Psi_{(\psi)}^{''2} \Theta_{(\theta)}^{''2} \frac{1}{4} B_1(\varphi) - \frac{1}{4\psi} \Psi_{(\psi)}^{''2} (\Theta_{(\theta)}^{'''})^2 \frac{1}{2} \bar{p} \bar{q} W_{(\varphi)}^{''2} \right. \\ \left. - \psi (\Psi_{(\psi)}^{'''})^2 \Theta_{(\theta)}^{''2} \frac{1}{2} \bar{p} \bar{q} V_{(\varphi)}^{''2} \right) e^{2imy} \quad (\text{II-108})$$

where

$$B_1(\varphi) = R'' P'' - \bar{q}^2 u'' v''$$

Using the following trigonometric identities

$$\Theta_{(\theta)}^{''2} = \cos^2 \nu \theta = \frac{1}{2} (1 + \cos 2\nu \theta)$$

$$(\Theta_{(\theta)}^{'''})^2 = \nu^2 \sin^2 \nu \theta = \frac{1}{2} (1 - \cos 2\nu \theta)$$

in Equation (II-108) gives:

$$B_{(2m)}^{(2)} = \left\{ \frac{1}{4} B_1(\varphi) \frac{1}{2} (1 + \cos 2\nu \theta) \bar{\Psi}_{(\psi)}^{''2} \right. \\ \left. - \frac{1}{2} \bar{p} \bar{q} W_{(\varphi)}^{''2} \frac{\nu^2}{2} (1 - \cos 2\nu \theta) \frac{1}{4\psi} \bar{\Psi}_{(\psi)}^{''2} \right. \\ \left. - \frac{1}{2} \bar{p} \bar{q} V_{(\varphi)}^{''2} \frac{1}{2} (1 + \cos 2\nu \theta) \psi (\bar{\Psi}_{(\psi)}^{'''})^2 \right\} e^{2imy} \quad (\text{II-108a})$$

To obtain the series expansion of $B_{(2m)}^{(2)}$ in terms of the eigenfunctions which are defined in Equation (II-107), each of the functions $(\Psi^{(1)}(\psi))^2$, $\psi(\Psi^{(1)}(\psi))^2$ and $\frac{1}{4\psi} \Psi^{(1)2}(\psi)$, which are contained in $B_{(2m)}^{(2)}$, must be expanded in two Dini-Series, once in terms of $J_0(S_{(0,q)} \sqrt{\frac{\psi}{\psi_w}})$ and once in terms of $J_{2\nu}(S_{(2\nu,q)} \sqrt{\frac{\psi}{\psi_w}})$. The exact form of these expansions is available in Appendix C. Substitution of these series expansions into Equation (II-108a) gives:

$$B_{(2m)}^{(2)} = e^{2imy} \left\{ \sum_{q=0}^{\infty} B_{(2m,0,q)}^{(2)} J_0(S_{(0,q)} \sqrt{\frac{\psi}{\psi_w}}) + \sum_{q=1}^{\infty} B_{(2m,2\nu,q)}^{(2)} \cos 2\nu\theta J_{2\nu}(S_{(2\nu,q)} \sqrt{\frac{\psi}{\psi_w}}) \right\} \quad (\text{II-108b})$$

where

$$B_{(2m,0,q)}^{(2)} = A_{(0,q)} \frac{1}{8} B_1(\varphi) - \frac{1}{4} \bar{p} \bar{q} V_{(1)}^{(1)2} (B_{(0,q)} + \nu^2 C_{(0,q)}) \frac{1}{\psi_w} \quad (\text{II-109a})^\#$$

and

$$B_{(2m,2\nu,q)}^{(2)} = A_{(2\nu,q)} \frac{1}{8} B_1(\varphi) - \frac{1}{4} \bar{p} \bar{q} V_{(1)}^{(1)2} (B_{(2\nu,q)} - \nu^2 C_{(2\nu,q)}) \frac{1}{\psi_w} \quad (\text{II-109b})$$

The relation $V^{(1)}(\varphi) = W^{(1)}(\varphi)$, which holds when the first order, axial component of vorticity is identically zero, was used in the derivation of the above equations.

The subscripts used in Equations (II-108b) and (II-109) are special cases of the more general combination of subscripts, i.e., $(km, n\nu, q)$, which indicate the time and space dependence of the eigenfunction which multiplies the subscripted coefficient. This form of

See Appendix C for the exact definitions of the constants $A_{(j,q)}$, $B_{(j,q)}$ and $C_{(j,q)}$, for $j = 0, 2\nu$, which appear in this and the following equations.

subscript notation will be used throughout the following analysis.

Using similar procedures, the eigenfunction expansion of the remainder of $B^{(2)}$ (i.e., $B_0^{(2)}$) as well as $C^{(2)}$, $D^{(2)}$, $E^{(2)}$ and $G^{(2)}$ can be easily derived. The details of this derivation will be omitted from this presentation and only the coefficients of the eigenfunctions, which are functions of φ , will now be presented.

$$B_{(0,0,q)}^{(2)} = \frac{1}{8} A_{(0,q)} (R'' P'' - \bar{q}^2 U'' V''^*) - \frac{1}{4\psi_w} (B_{(0,q)} + \nu^2 C_{(0,q)}) \bar{p} \bar{q} V'' V''^* \quad (\text{II-109c})$$

$$B_{(0,2\nu,q)}^{(2)} = \frac{1}{8} A_{(2\nu,q)} (R'' P'' - \bar{q}^2 U'' V''^*) - \frac{1}{4\psi_w} (B_{(2\nu,q)} - \nu^2 C_{(2\nu,q)}) \bar{p} \bar{q} V'' V''^* \quad (\text{II-109d})$$

$$A_{(2\nu,2\nu,q)}^{(2)} = \frac{1}{4} A_{(2\nu,q)} \left(\frac{1}{4} \frac{d\bar{q}^2}{d\varphi} \left(\frac{k-1}{\bar{c}^4} P''^2 + S''^2 \right) + R'' P'' - \frac{1}{2} (\bar{q}^2 U''^2)' \right) - \frac{1}{4\psi_w} (B_{(2\nu,q)} - \nu^2 C_{(2\nu,q)}) ((\bar{p} \bar{q})' V''^2 + 2 \bar{p} \bar{q} V'' V''^*) \quad (\text{II-110a})$$

$$A_{(0,2\nu,q)}^{(2)} = \frac{1}{4} A_{(2\nu,q)} \left(\frac{1}{4} \frac{d\bar{q}^2}{d\varphi} \left(\frac{k-1}{\bar{c}^4} P'' P''^* + S'' S''^* \right) + R'' P''^* - \frac{1}{2} (\bar{q}^2 U'' U''^*)' \right) - \frac{1}{4\psi_w} (B_{(2\nu,q)} - \nu^2 C_{(2\nu,q)}) ((\bar{p} \bar{q})' V'' V''^* + \bar{p} \bar{q} (V'' V''^*)') \quad (\text{II-110b})$$

$$A_{(2m,0,\bar{g})}^{(2)'} = \frac{1}{4} A_{(0,\bar{g})} \left(\frac{1}{4} \frac{d\bar{g}^2}{d\varphi} \left(\frac{\delta-1}{\bar{c}^4} P^{(2)} + S^{(2)} \right) + R^{(1)} P^{(1)'} - \frac{1}{2} (\bar{g}^2 V^{(1)2})' \right) \\ - \frac{1}{4\psi} (B_{(0,\bar{g})} + \nu^2 C_{(0,\bar{g})}) \frac{1}{4} \left((\bar{P}\bar{g})' V^{(1)2} + 2\bar{P}\bar{g} V^{(1)} V^{(1)'} \right)$$

(II-110c)

$$A_{(0,0,\bar{g})}^{(2)'} = \frac{1}{4} A_{(0,\bar{g})} \left(\frac{1}{4} \frac{d\bar{g}^2}{d\varphi} \left(\frac{\delta-1}{\bar{c}^4} P^{(1)} P^{(1)*} + S^{(1)} S^{(1)*} \right) + R^{(1)} P^{(1)*'} - \frac{1}{2} (\bar{g}^2 V^{(1)} V^{(1)*})' \right) \\ - \frac{1}{4\psi} (B_{(0,\bar{g})} + \nu^2 C_{(0,\bar{g})}) \frac{1}{4} \left((\bar{P}\bar{g})' V^{(1)} V^{(1)*} + \bar{P}\bar{g} (V^{(1)} V^{(1)*})' \right)$$

(II-110d)

Examination of Equation (II-74) shows that $E^{(2)}$ and $D^{(2)}$ appear in $I^{(2)}$ in the following combination: $\bar{c}^2 (E^{(2)} + D^{(2)})$. Using the definitions of $E^{(2)}$ and $D^{(2)}$ as well as some first order relations it can be shown that

$$\bar{c}^2 (E^{(2)} + D^{(2)}) = -\bar{c}^2 \left(\bar{g}^2 \mathcal{E}^{(1)} \left(\frac{\pi^{(1)}}{\bar{c}^2} \right)_{\varphi} - \mathcal{K}^{(1)} (i\omega^{(1)} \mathcal{K}^{(1)} + \bar{g}^2 \mathcal{K}_{\varphi}^{(1)}) \right)$$

$$- 2\bar{P}\bar{g} \psi \eta^{(1)} \pi_{\varphi}^{(1)} - \frac{1}{2\psi} \bar{P}\bar{g} \mathcal{E}^{(1)} \pi_0^{(1)}$$

(II-111)

Using complex notation the above expression can be rewritten in the following form:

$$\begin{aligned}
 \bar{C}^2(E^{(2)} + D^{(2)}) = & -\frac{1}{2} \bar{C}^2 \left\{ \left(\bar{q}^2 U'' \left(\frac{P''}{\bar{C}^2} \right)' - R'' (i\omega'' R'' + \bar{q}^2 R''') \right) \Theta''^2 e^{2im_y} \right. \\
 & + \left(\bar{q}^2 \left(\frac{P''}{\bar{C}^2} \right) U''^* - R''^* (i\omega'' R'' + \bar{q}^2 R''') \right) \Theta'' \Theta''^* \left. \right\} \Psi_{(\psi)}^2 \\
 & - \bar{P} \bar{q} \psi (\Psi_{(\psi)}')^2 \left(\Theta''^2 V'' P'' e^{2im_y} + V'' P''^* \Theta'' \Theta''^* \right) \\
 & - \bar{P} \bar{q} \frac{1}{4\psi} \Psi_{(\psi)}^2 \left(V'' P'' (\Theta'')^2 e^{2im_y} + V'' P''^* \Theta'' \Theta''^* \right)
 \end{aligned}$$

(II-112)

Expansion of Equation (II-112) in terms of the transverse eigenfunctions leads to the derivation of the following coefficients:

$$\begin{aligned}
 \bar{C}^2(E_{(\varphi)}^{(2)} + D_{(\varphi)}^{(2)})_{(2m, 2\mu, \delta)} = & -\frac{1}{4} \left\{ \bar{C}^2 A_{(2\mu, \delta)} \left(\bar{q}^2 U'' \left(\frac{P''}{\bar{C}^2} \right)' - R'' (i\omega'' R'' \right. \right. \\
 & \left. \left. + \bar{q}^2 R''') \right) - \frac{1}{\psi_\omega} (B_{(2\mu, \delta)} - \nu^2 C_{(2\mu, \delta)}) 2\bar{P} \bar{q} V'' P'' \right\}
 \end{aligned}$$

(II-113a)

$$\begin{aligned}
 \bar{C}^2(E_{(\varphi)}^{(2)} + D_{(\varphi)}^{(2)})_{(0, 2\mu, \delta)} = & -\frac{1}{4} \left\{ \bar{C}^2 A_{(0, \delta)} \left(\bar{q}^2 \left(\frac{P''}{\bar{C}^2} \right)' U''^* - R''^* (i\omega'' R'' \right. \right. \\
 & \left. \left. + \bar{q}^2 R''') \right) - \frac{1}{\psi_\omega} (B_{(0, \delta)} - \nu^2 C_{(0, \delta)}) 2\bar{P} \bar{q} V'' P''^* \right\}
 \end{aligned}$$

(II-113b)

$$\begin{aligned}
 \bar{C}^2(E_{(\varphi)}^{(2)} + D_{(\varphi)}^{(2)})_{(2m, 0, \delta)} = & -\frac{1}{4} \left\{ \bar{C}^2 A_{(2m, \delta)} \left(\bar{q}^2 \left(\frac{P''}{\bar{C}^2} \right)' U'' - R'' (i\omega'' R'' \right. \right. \\
 & \left. \left. + \bar{q}^2 R''') \right) - \frac{1}{\psi_\omega} (B_{(2m, \delta)} + \nu^2 C_{(2m, \delta)}) 2\bar{P} \bar{q} V'' P'' \right\}
 \end{aligned}$$

(II-113c)

$$\begin{aligned} \bar{C}^2 (E_{(\varphi)}^{(2)} + D_{(\varphi)}^{(2)})_{(0,0,\delta)} = & -\frac{1}{4} \left\{ \bar{C}^2 A_{(0,\delta)} \left(\bar{q}^2 \left(\frac{P''}{\bar{c}^2} \right) U''^* - R''^* (i\omega'' R'' + \bar{q}^2 R''') \right) \right. \\ & \left. - \frac{1}{\psi_0} (B_{(0,\delta)} + \nu^2 C_{(0,\delta)}) 2\bar{P}\bar{q} V''' P''^* \right\} \end{aligned}$$

(II-113d)

Finally the expansion of Equation (II-104) yields the following coefficients:

$$G_{(2m,2\nu,\delta)}^{(2)} = -\frac{1}{8} \bar{C}^2 A_{(2\nu,\delta)} \left(\frac{\delta}{\bar{c}^4} P''^2 - R''^2 \right) \quad (\text{II-114a})$$

$$G_{(0,2\nu,\delta)}^{(2)} = -\frac{1}{8} \bar{C}^2 A_{(2\nu,\delta)} \left(\frac{\delta}{\bar{c}^4} P'' P''^* - R'' R''^* \right) \quad (\text{II-114b})$$

$$G_{(2m,0,\delta)}^{(2)} = -\frac{1}{8} \bar{C}^2 A_{(0,\delta)} \left(\frac{\delta}{\bar{c}^4} P''^2 - R''^2 \right) \quad (\text{II-114c})$$

$$G_{(0,0,\delta)}^{(2)} = -\frac{1}{8} A_{(0,\delta)} \left(\frac{\delta}{\bar{c}^4} P'' P''^* - R'' R''^* \right) \quad (\text{II-114d})$$

Before proceeding with the expansion of $I^{(2)}$ it will be necessary to solve Equation (II-56) in which $S^{(2)}$ is the only unknown. Using the available eigenfunction expansions as an example it is assumed that the second order entropy, $S^{(2)}$, can be written in the following form:

$$S^{(2)} = \sum_{g=1}^{\infty} \left\{ S_{(\lambda m, 2\nu, g)}^{(2)} e^{2im\psi} + S_{(0, 2\nu, g)}^{(2)} \right\} \cos 2\nu\theta J_{2\nu}(S_{\lambda\nu, g} \sqrt{\frac{\psi}{\psi_w}}) \\ + \sum_{g=0}^{\infty} \left\{ S_{(\lambda m, 0, g)}^{(2)} e^{2im\psi} + S_{(0, 0, g)}^{(2)} \right\} J_0(S_{\lambda\nu, g} \sqrt{\frac{\psi}{\psi_w}}) \quad (\text{II-115})$$

where the coefficients $S_{(km, n\nu, q)}^{(2)}$ are still unknown. Substitution of $S^{(2)}$, as given above, as well as the eigenfunction expansion of $D^{(2)\#}$ into Equation (II-56) and separation of the variables in the resulting equation yields the following relation for each of the coefficients which appear in the expansion of $S^{(2)}$:

$$S_{(km, n\nu, g)}^{(2)} = f_0^{(km)}(\psi) \left\{ \int_0^\psi \frac{D_{(km, n\nu, g)}^{(2)}(\varphi)}{\bar{g}^2 f_0^{(km)}(\varphi)} d\varphi + \sigma_{(km, n\nu, g)}^{(2)} \right\} \quad (\text{II-116})$$

where

$$\sigma_{(km, n\nu, g)}^{(2)} = S_{(km, n\nu, g)}^{(2)} \quad (\text{II-117})$$

Finally to obtain the expansion form of $H^{(2)}$ the eigenfunction expansions of $A_\varphi^{(2)}$ and $S^{(2)}$ are substituted into Equation (II-60) yielding the following coefficients for the expansion of $H^{(2)}$:

$$H_{(km, n\nu, g)}^{(2)} = \tilde{H}_{(km, n\nu, g)}^{(2)} + \sigma_{(km, n\nu, g)}^{(2)} f_1^{(km)}(\psi) \quad (\text{II-118a})$$

[#] The eigenfunction expansion of $D^{(2)}$ can be easily obtained by a straight forward application of the methods that were used in the expansion of expressions that appear in $I^{(2)}$.

where

$$\begin{aligned} \tilde{H}_{(km, n, \nu, \xi)}^{(2)}(\varphi) &= \frac{1}{2} \int_0^\varphi \left(\frac{d\bar{g}^2}{d\varphi'} f_0^{(km)}(\varphi') \int_0^{\varphi'} \frac{D_{(km, n, \nu, \xi)}^{(2)}(\varphi'')}{\bar{g}^2(\varphi'') f_0^{(km)}(\varphi'')} d\varphi'' \right) d\varphi' \\ &+ \int_0^\varphi \frac{dA_{(km, n, \nu, \xi)}^{(2)}(\varphi')}{d\varphi'} d\varphi' \end{aligned} \quad (\text{II-119})$$

and $f_1^{(km)}$ is defined in Equation (II-77). Substitution of the eigenfunction expansions of $\bar{c}^2(E^{(2)} + D^{(2)})$, $G^{(2)}$, $B^{(2)}$ and $H^{(2)}$ into Equation (II-74) gives:

$$\begin{aligned} I_S^{(2)} &= \sum_{q=1}^{\infty} \left\{ I_{(2m, 2\nu, \xi)_S}^{(2)}(\varphi) e^{2im\gamma} + I_{(0, 2\nu, \xi)_S}^{(2)}(\varphi) \right\} \cos 2\nu\theta J_{2\nu}(S_{(2\nu, \xi)} \sqrt{\frac{\psi'}{\psi_w}}) \\ &+ \sum_{q=0}^{\infty} \left\{ I_{(2m, 0, \xi)_S}^{(2)}(\varphi) e^{2im\gamma} + I_{(0, 0, \xi)_S}^{(2)}(\varphi) \right\} J_0(S_{(0, \xi)} \sqrt{\frac{\psi'}{\psi_w}}) \end{aligned} \quad (\text{II-120})$$

where

$$\begin{aligned} I_{(km, n, \nu, \xi)_S}^{(2)}(\varphi) &= \bar{c}^2(E_{(\varphi)}^{(2)} + D_{(\varphi)}^{(2)})_{(km, n, \nu, \xi)} - \bar{c}^2 \bar{g}^2 \left(\frac{G_{(km, n, \nu, \xi)}^{(2)}(\varphi) + \tilde{H}_{(km, n, \nu, \xi)}^{(2)}(\varphi)}{\bar{c}^2} \right)' \\ &- ikm\omega^{(0)} \left(G_{(km, n, \nu, \xi)}^{(2)}(\varphi) + \tilde{H}_{(km, n, \nu, \xi)}^{(2)}(\varphi) \right) + \bar{c}^2 \bar{g}^2 f_0^{(km)} \frac{S_{(n, \nu, \xi)}^2}{2\psi_w} \int_0^\varphi \frac{B_{(km, n, \nu, \xi)}^{(2)}(\varphi') - \tilde{H}_{(km, n, \nu, \xi)}^{(2)}(\varphi')}{\bar{g}^2 f_0^{(km)}(\varphi')} d\varphi' \\ &+ C_{(km, n, \nu, \xi)}^{(2)} \bar{g}^2 \bar{c}^2 \frac{S_{(n, \nu, \xi)}^2}{2\psi_w} f_0^{(km)} - \sigma_{(km, n, \nu, \xi)}^{(2)} \bar{c}^2 \left(\bar{g}^2 \frac{S_{(n, \nu, \xi)}^2}{2\psi_w} f_2^{(km)} \right. \\ &\left. + \bar{g}^2 f_0^{(km)} f_3^{(km)} \right) \end{aligned} \quad (\text{II-121})$$

and

$$C_{(km, n, \zeta)}^{(2)} = V_{(km, n, \zeta)}^{(2)} - \Phi_{(km, n, \zeta)}^{(2)}$$

(II-122)

can be shown[#] to be related to the coefficients, evaluated at the throat, which appear in the eigenfunction expansion of the second order vorticity.

In order to obtain the complete solution of the second order equations, the remainder of the second order variables ($S^{(2)}$ has already been determined) are assumed to have the following series expansions:^{##}

$$F(\varphi, \psi, \theta, y) = \sum_{k=0,2}^{\infty} e^{ikmy} \left\{ \sum_{q=0}^{\infty} \Phi_{(km, q, \zeta)}^{(2)} \bar{J}_0(S_{(0, q, \zeta)}) \sqrt{\frac{\psi}{\psi_w}} \right. \\ \left. + \sum_{q=1}^{\infty} \Phi_{(km, 2q, \zeta)}^{(2)} \cos 2q\theta \bar{J}_{2q}(S_{(2q, q, \zeta)}) \sqrt{\frac{\psi}{\psi_w}} \right\}$$

(II-123a)

$$\Xi^{(2)}(\varphi, \psi, \theta, y) = \sum_{k=0,2}^{\infty} e^{ikmy} \left\{ \sum_{q=0}^{\infty} U_{(km, q, \zeta)}^{(2)} \bar{J}_0(S_{(0, q, \zeta)}) \sqrt{\frac{\psi}{\psi_w}} \right. \\ \left. + \sum_{q=1}^{\infty} U_{(km, 2q, \zeta)}^{(2)} \cos 2q\theta \bar{J}_{2q}(S_{(2q, q, \zeta)}) \sqrt{\frac{\psi}{\psi_w}} \right\}$$

(II-123b)

[#] For verification of this statement, see the derivation given in Appendix E.

^{##} In addition to the terms appearing in these series expansions, the second order variables have another component which has the same transverse and time dependence as the first order solution. This term results from the presence of terms proportional to $\omega^{(1)}$ in $I^{(2)}$. The derivation of these terms as well as the resulting additional components of the solution will be discussed separately.

$$\begin{aligned} \tilde{\gamma}_{(\varphi, \psi, \theta, y)}^{(2)} = \sum_{k=0,2} e^{ikmy} \left\{ \sum_{q=0}^{\infty} V_{(km,0,q)}^{(2)} J_0(S_{(0,q)} \sqrt{\frac{\psi}{\psi_w}}) \right. \\ \left. + \sum_{q=1}^{\infty} V_{(km,2q,q)}^{(2)} \cos 2\nu\theta J_{2\nu}(S_{(2q,q)} \sqrt{\frac{\psi}{\psi_w}}) \right\} \end{aligned} \quad (\text{II-123c})$$

$$\begin{aligned} \tilde{\xi}_{(\varphi, \psi, \theta, y)}^{(2)} = \sum_{k=0,2} e^{ikmy} \left\{ \sum_{q=0}^{\infty} W_{(km,0,q)}^{(2)} J_0(S_{(0,q)} \sqrt{\frac{\psi}{\psi_w}}) \right. \\ \left. + \sum_{q=1}^{\infty} W_{(km,2q,q)}^{(2)} \cos 2\nu\theta J_{2\nu}(S_{(2q,q)} \sqrt{\frac{\psi}{\psi_w}}) \right\} \end{aligned} \quad (\text{II-123d})$$

$$\begin{aligned} \Pi_{(\varphi, \psi, \theta, y)}^{(2)} = \sum_{k=0,2} e^{ikmy} \left\{ \sum_{q=0}^{\infty} P_{(km,0,q)}^{(2)} J_0(S_{(0,q)} \sqrt{\frac{\psi}{\psi_w}}) \right. \\ \left. + \sum_{q=1}^{\infty} P_{(km,2q,q)}^{(2)} \cos 2\nu\theta J_{2\nu}(S_{(2q,q)} \sqrt{\frac{\psi}{\psi_w}}) \right\} \end{aligned} \quad (\text{II-123e})$$

$$\begin{aligned} \kappa_{(\varphi, \psi, \theta, y)}^{(2)} = \sum_{k=0,2} e^{ikmy} \left\{ \sum_{q=0}^{\infty} R_{(km,0,q)}^{(2)} J_0(S_{(0,q)} \sqrt{\frac{\psi}{\psi_w}}) \right. \\ \left. + \sum_{q=1}^{\infty} R_{(km,2q,q)}^{(2)} \cos 2\nu\theta J_{2\nu}(S_{(2q,q)} \sqrt{\frac{\psi}{\psi_w}}) \right\} \end{aligned} \quad (\text{II-123f})$$

Substitution of Equations (II-123) and (II-115) into Equation (II-61) and separation of variables yields the following ordinary differential equation for each of the coefficients $\Phi_{(km,n\nu,q)}^{(2)}$ of $F^{(2)}$:

$$\begin{aligned} \bar{q}^2 (\bar{c}^2 - \bar{q}^2) \frac{d^2}{d\varphi^2} \Phi_{(km,n\nu,q)}^{(2)} - \bar{q}^2 \left(2ikm\omega^{(0)} + \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} \right) \frac{d}{d\varphi} \Phi_{(km,n\nu,q)}^{(2)} \\ + \left(k^2 m^2 \omega^{(0)2} - \frac{\bar{q}^2}{\bar{c}^2} \frac{\delta-1}{2} \frac{d\bar{q}^2}{d\varphi} ikm\omega^{(0)} - \bar{p}\bar{q}\bar{c}^2 \frac{S_{(n\nu,q)}^2}{2\psi_w} \right) \Phi_{(km,n\nu,q)}^{(2)} = I_{(km,n\nu,q)}^{(2)} \end{aligned} \quad (\text{II-124})$$

For further reference it will be convenient to rewrite Equation (II-124) in the following form:

$$\begin{aligned} \mathcal{L}_{(km, n\nu, \delta)}^{(2)} (\bar{\Phi}_{(km, n\nu, \delta)}^{(2)}) &= -\sigma_{(km, n\nu, \delta)}^{(2)} I_{e(km, n\nu, \delta)}^{(2)} + C_{(km, n\nu, \delta)}^{(2)} I_{v(km, n\nu, \delta)}^{(2)} \\ &+ I_{N(km, n\nu, \delta)}^{(2)} \end{aligned} \quad (\text{II-125})$$

where

$\mathcal{L}_{(km, n\nu, \delta)}^{(2)} (\bar{\Phi}_{(km, n\nu, \delta)}^{(2)})$ represents the left-hand side of Equation (II-124) and

$$I_{e(km, n\nu, \delta)}^{(2)} = \bar{C}^2 \left(\bar{p} \bar{q} \frac{S_{(n\nu, \delta)}^2}{2\psi_\omega} f_2^{(km)} + \bar{q}^{-2} f_0^{(km)} f_3^{(km)} \right) \quad (\text{II-126a})$$

$$I_{v(km, n\nu, \delta)}^{(2)} = \bar{p} \bar{q} \bar{C}^2 \frac{S_{(n\nu, \delta)}^2}{2\psi_\omega} f_0^{(km)} \quad (\text{II-126b})$$

$$\begin{aligned} I_{N(km, n\nu, \delta)}^{(2)} &= \bar{C}^2 (E_{(\varphi)}^{(2)} + D_{(\varphi)}^{(2)})_{(km, n\nu, \delta)} - \bar{C}^2 \bar{q}^{-2} \left(\frac{G_{(km, n\nu, \delta)}^{(2)} + \tilde{H}_{(km, n\nu, \delta)}^{(2)}}{\bar{C}^2} \right)' \\ &- i k m \omega^{(n)} \left(G_{(km, n\nu, \delta)}^{(2)} + \tilde{H}_{(km, n\nu, \delta)}^{(2)} \right) + \bar{C}^2 \bar{p} \bar{q} f_0^{(km)} \frac{S_{(n\nu, \delta)}^2}{2\psi_\omega} \int_0^\varphi \frac{B_{(km, n\nu, \delta)}^{(2)} - \tilde{H}_{(km, n\nu, \delta)}^{(2)}}{\bar{q}^2 f_0^{(km)}} d\phi' \end{aligned} \quad (\text{II-126c})$$

Above expressions were obtained from $I_{(km, n\nu, q)}^{(2)}$ which is defined in Equation (II-121).

Assuming that the solution of Equation (II-124) is available, we can proceed to solve for the second order components of vorticity, velocity, pressure and density.

Using Equations (II-123c) and (II-52) the second order radial component of velocity can be expressed in the following form:

$$\eta^{(2)} = \sum_{k=0,2} e^{ikmy} \left\{ \sum_{g=0} V_{(k,m,0,g)}^{(2)} \frac{d}{d\psi} J_0(S_{(0,g)} \sqrt{\frac{\psi}{\psi_w}}) \right. \\ \left. + \sum_{g=1}^{\infty} V_{(k,m,2,g)}^{(2)} \cos 2\nu\theta \frac{d}{d\psi} J_{2\nu}(S_{(2,g)} \sqrt{\frac{\psi}{\psi_w}}) \right\} \quad (\text{II-127})$$

Similarly using Equations (II-123d) and (II-53) the expansion of the tangential component of velocity is:

$$\zeta^{(2)} = \sum_{k=0,2} e^{ikmy} \sum_{g=1}^{\infty} W_{(k,m,2,g)}^{(2)} \left(\frac{d}{d\theta} \cos 2\nu\theta \right) J_{2\nu}(S_{(2,g)} \sqrt{\frac{\psi}{\psi_w}}) \quad (\text{II-128})$$

It is important to note that the second order tangential component of velocity has no terms proportional to J_0 .

Using Equations (II-51), (II-123a) and (II-123b) it can be shown that

$$U_{(k,m,n,g)}^{(2)} = \frac{d}{d\varphi} \Phi_{(k,m,n,g)}^{(2)} \quad (\text{II-129})$$

Substitution of the eigenfunction expansions of $\eta^{(2)}$, $\zeta^{(2)}$, $\xi^{(2)}$ into Equation (II-63), gives the eigenfunction expansion of the vorticity

in which each of the φ dependent coefficients has the following form:

$$\begin{aligned}
 (\nabla \times \vec{q}^{(2)})_{\vec{r}}(\varphi) &= \frac{\bar{q}}{r} \left(\Phi_{(km, n\nu, q)}^{(2)'} - V_{(km, n\nu, q)}^{(2)'} \right) \vec{e}_\psi \\
 &+ \bar{p}\bar{q}r \left(V_{(km, n\nu, q)}^{(2)'} - \Phi_{(km, n\nu, q)}^{(2)'} \right) \vec{e}_\theta
 \end{aligned}
 \tag{II-130}$$

As in the expansion of $\mathfrak{S}^{(2)}$, the tangential component of the vorticity has no terms which are proportional to J_0 . The expressions inside the square brackets in Equation (II-130) can be determined in terms of known quantities by expanding Equations (II-67) and (II-68). Substitution of Equations (II-127), (II-128) and the series expansions of $B^{(2)}$, $H^{(2)}$ and $F^{(2)}$ into Equation (II-58) gives:

$$\begin{aligned}
 V_{(km, n\nu, q)}^{(2)} &= f_0^{(km)}(\varphi) \left\{ \int_0^\varphi \frac{B_{(km, n\nu, q)}^{(2)} - \tilde{H}_{(km, n\nu, q)}^{(2)}}{\bar{q}^2 f_0^{(km)}} d\varphi' + \frac{\Phi_{(km, n\nu, q)}^{(2)}}{f_0^{(km)}(\varphi)} \right\} \\
 &+ f_0^{(km)} \left(V_{(0)}^{(2)} - \Phi_{(0)}^{(2)} \right) - \sigma_{(km, n\nu, q)}^{(2)} f_2^{(km)}
 \end{aligned}
 \tag{II-131}$$

Since $B^{(2)} = C^{(2)}$ it can be shown by use of Equation (II-59) that the following relation holds:

$$V_{(km, n\nu, q)}^{(2)}(\varphi) = W_{(km, n\nu, q)}^{(2)}(\varphi)
 \tag{II-132}$$

which is analogous with the results obtained in the first order analysis.

Substitution of the expansions of $F^{(2)}$ and $S^{(2)}$ into Equation (II-46a) and separating the variables gives:

$$P_{(km, n\nu, g)}^{(2)} = \int_0^{\varphi} \frac{1}{2} \frac{d\bar{q}^2}{d\varphi} S_{(km, n\nu, g)}^{(2)} d\varphi + A_{(km, n\nu, g)}^{(2)} - i k m \omega^{(1)} \Phi_{(km, n\nu, g)}^{(2)} - \bar{q}^2 \frac{d}{d\varphi} \Phi_{(km, n\nu, g)}^{(2)} \quad (\text{II-133})$$

Once $\Pi^{(2)}$ and $S^{(2)}$ are known, the coefficients of $\Phi^{(2)}$ can be determined from Equation (II-50):

$$R_{(km, n\nu, g)}^{(2)} = \frac{1}{c^2} \left(G_{(km, n\nu, g)}^{(2)} + P_{(km, n\nu, g)}^{(2)} \right) - S_{(km, n\nu, g)}^{(2)} \quad (\text{II-134})$$

The solution of Equation (II-125) for $\Phi_{(km, n\nu, q)}^{(2)}$ as well as the determination of $\sigma_{(km, n\nu, q)}^{(2)}$, $V_{(km, n\nu, q)}^{(2)}$ and $\Phi_{(km, n\nu, q)}^{(2)}$, whose knowledge is necessary for the determination of the coefficients which are present in the eigenfunction expansions of the second order variables, will be presented in other sections of this chapter.

Travelling-Wave Motion

The analysis performed so far has been concerned with standing-wave motion only. To consider the case of travelling-wave motion we must return to Equations (II-97) through (II-102) and obtain the eigenfunction expansions of these expressions which are applicable to this case. In this case, substituting $\Theta_{(o)}^{(1)} = e^{i\nu\varphi}$ into the complex expressions for $A_{\varphi}^{(2)}$ and following the same procedure as the one used in the standing-wave

analysis gives the following series expansion:

$$A_{\varphi}^{(2)} = \sum_{q=1}^{\infty} 2A_{(2m, 2\nu, q)}^{(2)'} e^{2i(\nu\theta + m\gamma)} J_{2\nu}(S_{(2\nu, q)} \sqrt{\frac{\psi'}{\psi_w}}) \\ + \sum_{q=0}^{\infty} 2A_{(0, 0, q)}^{(2)'} J_0(S_{(0, q)} \sqrt{\frac{\psi'}{\psi_w}})$$

(II-135)

where the expressions for $A_{(2m, 2\nu, q)}^{(2)'}$ and $A_{(0, 0, q)}^{(2)'}$ are given in

Equations (II-110a) and (II-110d). Similar expansions can be obtained for $B^{(2)}$, $C^{(2)}$, $\bar{c}^2(E^{(2)} + D^{(2)})$ and $G^{(2)}$. A comparison between the eigenfunction expansions for the two cases shows that in the case of travelling-wave motion the expansions do not contain any coefficients in which the subscript $(km, n\nu, q)$ contains a zero for either km or $n\nu$. The coefficients which appear in these expansions have the same definitions and are twice as large as the corresponding coefficients in the case of standing-wave motion. Using the expansions of the terms which appear in the inhomogeneous part as an example, the second order variables describing the travelling-wave motion are assumed to have an eigenfunction expansion of the following form:

$$\tilde{X} = \sum_{q=1}^{\infty} \tilde{X}_{(2m, 2\nu, q)}^{(2)} e^{2i(\nu\theta + m\gamma)} J_{2\nu}(S_{(2\nu, q)} \sqrt{\frac{\psi'}{\psi_w}}) \\ + \sum_{q=0}^{\infty} \tilde{X}_{(0, 0, q)}^{(2)} J_0(S_{(0, q)} \sqrt{\frac{\psi'}{\psi_w}})$$

(II-136)

where $\tilde{X}_{(km,n\nu,q)}^{(2)}$ is a dummy variable which can be replaced by anyone of the following quantities $\Phi_{(km,n\nu,q)}^{(2)}$, $U_{(km,n\nu,q)}^{(2)}$, $V_{(km,n\nu,q)}^{(2)}$, $W_{(km,n\nu,q)}^{(2)}$, $P_{(km,n\nu,q)}^{(2)}$, $R_{(km,n\nu,q)}^{(2)}$ and $S_{(km,n\nu,q)}^{(2)}$. Using the suggested series expansions for the inhomogeneous parts of the equations as well as for the second order unknowns and following the same procedure as in the analysis of the standing-wave motion would yield, with one exception, the same results as in the latter case. The exception is that in the case of travelling-wave motion, the variables $A'_{(km,n\nu,q)}(\varphi)$, $B_{(km,n\nu,q)}(\varphi)$, $\bar{c}^2(E_{(km,n\nu,q)}^{(2)} + D_{(km,n\nu,q)}^{(2)})$, and $G_{(km,n\nu,q)}^{(2)}$ whose definitions were given before, must be multiplied by two wherever they appear. In view of the remarks made above, the derivation of the equations and solutions describing the travelling wave motion will not be given here.

Irrotational Nozzle Flow

In this section the equations which describe the second order irrotational flow conditions inside the converging section of a nozzle will be discussed. The knowledge of the solutions of these equations and the resulting admittance relation is necessary for the analysis of problems in which the flow inside the combustion chamber is assumed to be irrotational. The relations derived in this section will be used, later on in this thesis, in a specific problem in which the stability of finite amplitude pressure waves inside the combustion chamber of liquid propellant rocket engines will be investigated.

Using the first order relations for irrotational flow in Equations (II-43b) and (II-43c) yields the following results:

$$A_{\varphi}^{(2)} = \frac{1}{2} \left(\frac{\pi''^2}{\bar{c}^2} \right)_{\varphi} - \frac{1}{2} (\bar{q}^2 \bar{\zeta}''^2)_{\varphi} - (\psi \bar{p} \bar{q} \eta''^2)_{\phi} - \left(\frac{1}{4\psi} \bar{p} \bar{q} \bar{\zeta}''^2 \right)_{\phi} \quad (\text{II-137})$$

and

$$B_{\psi}^{(2)} = \frac{1}{2} \left(\frac{\pi^{(2)}}{\bar{c}^2} \right)_{\psi} - \frac{1}{2} (\bar{q}^2 \xi^{(2)})_{\psi} - (\psi \bar{p} \bar{q} \eta^{(2)})_{\psi} - \left(\frac{1}{4\psi} \bar{p} \bar{q} \xi^{(2)} \right)_{\psi} \quad (\text{II-138})$$

Differentiation of Equations (II-137) and (II-138) gives:

$$A_{\phi\psi}^{(2)} = B_{\psi\phi}^{(2)} \quad (\text{II-139})$$

which is one of the conditions which must be satisfied whenever the flow is irrotational. From previous discussions and Appendix E it can be shown that whenever the flow is irrotational (and hence homentropic) the following relations:

$$S^{(1)} = S_{(km, n, q)}^{(2)} = \sigma^{(1)} = \sigma_{(km, n, q)}^{(2)} = C_1^{(1)} = C_{(km, n, q)}^{(2)} = 0 \quad (\text{II-140})$$

must be satisfied. Substitution of Equation (II-140) into Equation (II-60) and integration of Equations (II-137) and (II-138) gives:#

$$H^{(2)} = A^{(2)} = B^{(2)} \quad (\text{II-141})$$

Finally substitution of Equations (II-140) and (II-141) into Equation (II-74) gives:

$$I_{\text{irrotational}}^{(2)} = \bar{c}^2 \left\{ E^{(2)} - \bar{q}^2 \left(\frac{G^{(2)} + A^{(2)}}{\bar{c}^2} \right)_{\phi} \right\} - i k m \omega^{(1)} (G^{(2)} + A^{(2)}) \quad (\text{II-142})$$

which is the inhomogeneous part of Equation (II-61) for the case of irrotational flow. A comparison between Equations (II-74) and (II-142) shows that the latter is considerably simpler. The resulting simplification

The functions of the integrations will be taken to be identically zero. Following a different procedure it can be shown that these constants are "absorbed" by the arbitrary functions which are included in the definitions of $F^{(j)}$ and $\tilde{\eta}^{(j)}$.

is not sufficient, however, to avoid the use of series expansions in the solution of this problem. Following the same procedure as before, the eigenfunction expansion of $I_{\text{irrot}}^{(2)}$ for the case of standing wave motion gives the following result:[#]

$$I_{\text{irrot}_s}^{(2)} = \sum_{q=0}^{\infty} \left\{ \frac{1}{2} A_{(0,q)} (X(\varphi) e^{2imy} + \beth(\varphi)) + \frac{1}{2\psi_w} (B_{(0,q)} + \nu^2 C_{(0,q)}) (\lambda(\varphi) e^{2imy} + \daleth(\varphi)) \right\} J_0(S_{(0,q)} \sqrt{\frac{\psi}{\psi_w}}) + \sum_{q=1}^{\infty} \left\{ \frac{1}{2} A_{(2,q)} (X(\varphi) e^{2imy} + \beth(\varphi)) + \frac{1}{2\psi_w} (B_{(2,q)} - \nu^2 C_{(2,q)}) (\lambda(\varphi) e^{2imy} + \daleth(\varphi)) \right\} \quad (\text{II-143})$$

When travelling-wave motion is being considered the inhomogeneous part of Equation (II-61) can be written in the following form:

$$I_{\text{irrot}_T}^{(2)} = \sum_{q=0}^{\infty} \left(A_{(0,q)} \beth(\varphi) + \frac{1}{\psi_w} (B_{(0,q)} + \nu^2 C_{(0,q)}) \daleth(\varphi) \right) J_0(S_{(0,q)} \sqrt{\frac{\psi}{\psi_w}}) + \sum_{q=1}^{\infty} \left(A_{(2,q)} X(\varphi) + \frac{1}{\psi_w} (B_{(2,q)} - \nu^2 C_{(2,q)}) \lambda(\varphi) \right) e^{2i(\nu\theta + my)} J_{2\nu}(S_{(2,q)} \sqrt{\frac{\psi}{\psi_w}}) \quad (\text{II-143a})$$

where

$$X(\varphi) = \frac{1}{2} \bar{C}^2 \left(R^{(1)} (i\omega^{(1)} R^{(1)} + \bar{q}^2 R^{(1)'}) - \bar{q}^2 U^{(1)} \left(\frac{P^{(1)}}{\bar{C}^2} \right)' - \bar{q}^2 \left(\frac{2-\delta}{2} \frac{P^{(1)2}}{\bar{C}^4} - \frac{1}{2} \frac{\bar{q}^2}{\bar{C}^2} U^{(1)2} \right)' - \frac{2i\omega^{(1)}}{\bar{C}^2} \left(\frac{2-\delta}{2} \frac{P^{(1)2}}{\bar{C}^4} - \frac{1}{2} \bar{q}^2 U^{(1)2} \right) \right) \quad (\text{II-144})$$

[#] The symbols X (alef), \beth (beit), λ (gimel) and \daleth (dalet) which are used in the following equation are the first four letters of the Hebrew alphabet.

$$\begin{aligned} \Gamma(\varphi) = & \frac{1}{2} \bar{C}^2 \left\{ R^{(n)*} (i\omega^{(n)} R^{(n)} + \bar{q}^2 R^{(n)})' - \bar{q}^2 U^{(n)*} \left(\frac{P^{(n)}}{\bar{C}^2} \right)' \right. \\ & \left. - \bar{q}^2 \left(\frac{2-i}{2} \frac{P^{(n)} P^{(n)*}}{\bar{C}^4} - \frac{1}{2} \frac{\bar{q}^2}{\bar{C}^2} U^{(n)} U^{(n)*} \right) \right\} \end{aligned} \quad (\text{II-145})$$

$$\lambda(\varphi) = \frac{1}{2} \bar{C}^2 \bar{q}^2 \left(\frac{1}{\bar{C}^2} \bar{P} \bar{q} V^{(n2)} \right)' + i\omega^{(n)} \bar{P} \bar{q} V^{(n2)} - \bar{P} \bar{q} V^{(n)} P^{(n)} \quad (\text{II-146})$$

$$T(\varphi) = \frac{1}{2} \bar{C}^2 \bar{q}^2 \left(\frac{1}{\bar{C}^2} \bar{P} \bar{q} V^{(n)} V^{(n)*} \right)' - \bar{P} \bar{q} V^{(n)} P^{(n)*} \quad (\text{II-147})$$

and the definitions of the constants $A_{(j,q)}$, $B_{(j,q)}$ and $C_{(j,q)}$ for $j = 0, 2V$ are available in Appendix C. Using the results derived in this section and repeating the analysis performed in pages (51) through (62) of the previous section would provide the differential equations and the form of the solutions which describe the second order irrotational flow.

Third Order Analysis

To obtain the solutions of the third order equations their inhomogeneous parts as given by Equations (II-44a) through (II-44g) must be rewritten in complex form and then expanded in terms of the appropriate eigenfunctions. Since the inhomogeneous parts of the third order equations contain products of first order and second order quantities and the latter are available in series forms, their manipulation and expansion are

considerably more involved than the manipulation and expansion of the corresponding expressions in the second order equations. In addition the eigenfunctions used in the expansions of the third order variables are entirely different than the eigenfunctions used in the second order expansions. With the aforesaid exceptions, once the dependent variables and the inhomogeneous parts of the third order equations have been expanded and substituted into the appropriate equations, the solution of the resulting equations for the coefficients which appear in expansion of the third order unknowns can be obtained in exactly the same manner as in the second order analysis. Consequently the general solution of the third order equations, which is very long, will not be given here. Instead the special case in which the flow in the subsonic section of the nozzle is assumed to be irrotational will be considered in detail. The results obtained in this analysis will be applied to the solution of the specific problem in which the stability of finite amplitude, irrotational pressure waves inside the combustion chamber of liquid propellant rocket engines will be investigated.

Using Equations (II-41), (II-42), (II-43f) and (II-60) and assuming that the first and second order flows are irrotational, it can be shown that

$$A_{\phi}^{(3)} = H_{\phi}^{(3)} = \left(-\frac{1}{6} (\gamma^2 - 3\gamma + 2) \bar{q}^2 + \frac{1}{3} (\gamma - 2) \right) (\kappa^{(3)})_{\phi} + (\bar{c}^2 \kappa^{(1)} \kappa^{(2)})_{\phi} - L_{\phi} \quad (\text{II-148})$$

$$B_{\psi}^{(3)} = \left(-\frac{1}{6} (\gamma^2 - 3\gamma + 2) + \frac{1}{3} (\gamma - 2) \right) (\kappa^{(3)})_{\psi} + (\bar{c}^2 \kappa^{(1)} \kappa^{(2)})_{\psi} - L_{\psi}$$

where L is defined in Equation (II-44g) (II-149)

Integrating Equations (II-148) and (II-149) with respect to φ and ψ gives:[#]

$$H^{(3)} = A^{(3)} = B^{(3)} \quad (\text{II-150})$$

Using a similar procedure it can be shown that

$$A^{(3)} = C^{(3)} \quad (\text{II-151})$$

Using Equations (II-150) and (II-151) and assuming that the flow is homentropic and that all three components of the third order vorticity are zero at the nozzle throat it can be easily shown, by use of Equations (II-69a) through (II-71c), that all three components of the third order vorticity are identically zero.

Assuming that the first and second order solutions, for the case of irrotational flow are completely known, we can proceed with the eigenfunction expansion of the inhomogeneous parts of the third order equations. Since detailed examples of such expansions for typical third order expressions are available in Appendix B, the expansion of the third order equations will not be covered here in detail. Using Equation (II-150) and collecting terms according to their dependence on time^{##} yields the following expression for the inhomogeneous part of Equation (II-61):

$$I_{(km)}^{(3)} \text{ irrot.} = \bar{c}^2 E_{(km)}^{(3)} - \bar{c}^2 \bar{q}^2 \left(\frac{G_{(km)}^{(3)} + A_{(km)}^{(3)}}{\bar{c}^2} \right) \varphi - i k m \omega^{(0)} (G_{(km)}^{(3)} + A_{(km)}^{(3)}) \quad (\text{II-152})$$

where $k = 1$ or 3 . In the following paragraphs the complete definitions of $E^{(3)}$, $A^{(3)}$ and $G^{(3)}$ which applies to the case of irrotational flow will be given:

[#] The result obtained here is quite general as it applies to both the travelling and the standing-wave motions. The constants that result from the integration of these equations are assumed to be "absorbed" by $F^{(3)}$ and $\tilde{\eta}^{(3)}$ which are defined, within an arbitrary constant, in Equations (II-51) and (II-52)

^{##} See discussion on page 28.

$$\begin{aligned}
 E^{(3)} = & \chi^{(1)} \left(\omega^{(0)}(\chi^{(2)})_y + \bar{q}^2(\chi^{(2)})_\varphi \right) + (\chi^{(2)} - \chi^{(1)2}) \left(\omega^{(0)}(\chi^{(1)})_y + \bar{q}^2(\chi^{(1)})_\varphi \right) \\
 & + \chi^{(1)} \xi^{(1)}(\chi^{(1)})_\varphi \bar{q}^2 + 2\bar{p}\bar{q} \chi^{(1)} \psi \eta^{(1)}(\chi^{(1)})_\psi + \frac{1}{2\psi} \bar{p}\bar{q} \chi^{(1)} \xi^{(1)}(\chi^{(1)})_\theta \\
 & - \bar{q}^2 \left(\xi^{(2)}(\chi^{(1)})_\varphi + \xi^{(1)}(\chi^{(2)})_\varphi \right) - 2\bar{p}\bar{q} \psi \left(\eta^{(2)}(\chi^{(1)})_\psi + \eta^{(1)}(\chi^{(2)})_\psi \right) \\
 & - \frac{1}{2\psi} \bar{p}\bar{q} \left(\xi^{(2)}(\chi^{(1)})_\theta + \xi^{(1)}(\chi^{(2)})_\theta \right)
 \end{aligned} \tag{II-153}$$

$$\begin{aligned}
 A^{(3)} = & \left(\bar{q}^2 \left(\frac{1}{2}(\delta-1) - \frac{1}{6}(\delta^2-1) \right) + \frac{1}{3}(\delta-2) \right) \chi^{(1)3} + \bar{c}^2 \chi^{(1)} \chi^{(2)} \\
 & - \left(\bar{q}^2 \xi^{(1)} \xi^{(2)} + 2\bar{p}\bar{q} \psi \eta^{(1)} \eta^{(2)} + \frac{1}{2\psi} \bar{p}\bar{q} \xi^{(1)} \xi^{(2)} \right)
 \end{aligned} \tag{II-154}$$

and

$$G^{(3)} = \frac{1}{3} \bar{c}^2 (\delta-1 - \frac{3}{2}\delta(\delta-1)) \chi^{(1)3} - \bar{c}^2 (\delta-1) \chi^{(1)} \chi^{(2)} \tag{II-155}$$

Substituting Expressions (II-153) through (II-155) into Equation (II-152) for $k = 1$ and $k = 3$ and expanding the resulting expression according to the procedure outlined in the examples of Appendix B yields the following series:

$$\begin{aligned}
 I_{int.}^{(3)} = & \sum_{q=1}^{\infty} \left\{ \left(I_{(3m, 3\nu, \xi')}^{(3)} e^{3imy} + I_{(m, 3\nu, \xi')}^{(3)} e^{imy} \right) \cos 3\nu\theta J_{3\nu} \left(S_{(3\nu, \xi')} \sqrt{\frac{\psi}{\psi_w}} \right) \right. \\
 & \left. + \left(I_{(3m, \nu, \xi')}^{(3)} e^{3imy} + I_{(m, \nu, \xi')}^{(3)} e^{imy} \right) \cos \nu\theta J_{\nu} \left(S_{(\nu, \xi')} \sqrt{\frac{\psi}{\psi_w}} \right) \right\} \tag{II-156}
 \end{aligned}$$

which applies to the case of standing-wave motion and where

$$\begin{aligned}
 I_{(3m, 3\nu, q')}^{(3)} = & \sum_{q=0}^{\infty} \left\{ (AA_1 + AA_7 + AA_9 + AA_{11} + AA_{13} + AA_{15} \right. \\
 & + AA_{18}) A_{(3\nu, q')}^{(2\nu, q')} + \frac{1}{\psi_w} \left((AA_3 + AA_{16} + AA_{19}) B_{(3\nu, q')}^{(2\nu, q')} + (AA_5 + AA_{17} + AA_{20}) C_{(3\nu, q')}^{(2\nu, q')} \right) \Big\} \\
 & + \frac{1}{\psi_w} (AA_4 D_{(3\nu, q')}^{(\nu, \nu)} + AA_6 M_{(3\nu, q')}^{(\nu, \nu)}) + N_{(3\nu, q')}^{(\nu, \nu)} (AA_2 + AA_{10} + AA_{14} \\
 & + AA_{14})
 \end{aligned} \tag{II-157}$$

$$\begin{aligned}
 I_{(m, 3\nu, q')}^{(3)} = & \sum_{q=0}^{\infty} \left\{ (BB_1 + BB_7 + BB_9 + BB_{11} + BB_{13} + BB_{15} \right. \\
 & + BB_{18}) A_{(3\nu, q')}^{(2\nu, q')} + \frac{1}{\psi_w} \left((BB_3 + BB_{16} + BB_{19}) B_{(3\nu, q')}^{(2\nu, q')} + (BB_5 \right. \\
 & + BB_{17} + BB_{20}) C_{(3\nu, q')}^{(2\nu, q')} \Big\} + (BB_4 D_{(3\nu, q')}^{(\nu, \nu)} + BB_6 M_{(3\nu, q')}^{(\nu, \nu)}) \frac{1}{\psi_w} + (BB_2 \\
 & + BB_8 + BB_{10} + BB_{12} + BB_{14}) N_{(3\nu, q')}^{(\nu, \nu)}
 \end{aligned} \tag{II-158}$$

$$\begin{aligned}
 I_{(3m, \nu, q')}^{(3)} = & \sum_{q=0}^{\infty} \left\{ (CC_1 + CC_{10} + CC_{13} + CC_{16} + CC_{19} + CC_{22} + CC_{28}) A_{(\nu, q')}^{(2\nu, q')} \right. \\
 & + (CC_2 + CC_{11} + CC_{14} + CC_{17} + CC_{20} + CC_{23} + CC_{29}) A_{(\nu, q')}^{(\nu, q')} + \frac{1}{\psi_w} \left((CC_4 + CC_{24} \right. \\
 & + CC_{30}) B_{(\nu, q')}^{(\nu, q')} + (CC_5 + CC_{25} + CC_{31}) B_{(\nu, q')}^{(2\nu, q')} + (CC_7 + CC_{26} + CC_{32}) C_{(\nu, q')}^{(2\nu, q')} \Big\} \\
 & + \frac{1}{\psi_w} (CC_6 D_{(\nu, q')}^{(\nu, \nu)} + CC_9 M_{(\nu, q')}^{(\nu, \nu)}) + (CC_3 + CC_{12} + CC_{15} + CC_{18} + CC_{21}) N_{(\nu, q')}^{(\nu, \nu)}
 \end{aligned} \tag{II-159}$$

$$\begin{aligned}
 I_{(m, \nu, q')}^{(3)} = & \sum_{q=0}^{\infty} \left\{ (DD_1 + DD_{10} + DD_{13} + DD_{16} + DD_{19} + DD_{22} + DD_{28}) A_{(\nu, \xi')}^{(2\nu, 8)} \right. \\
 & + (DD_2 + DD_{11} + DD_{14} + DD_{17} + DD_{20} + DD_{23} + DD_{29}) A_{(\nu, \xi')}^{(0, 8)} + \frac{1}{\psi} ((DD_4 + DD_{24} \\
 & + DD_{30}) B_{(\nu, \xi')}^{(2\nu, 8)} + (DD_5 + DD_{25} + DD_{31}) B_{(\nu, \xi')}^{(0, 8)} + (DD_7 + DD_{26} + DD_{32}) C_{(\nu, \xi')}^{(2\nu, 8)} \Big) \Big\} \\
 & + \frac{1}{\psi} (DD_6 D_{(\nu, \xi')}^{(\nu, \nu)} + DD_9 M_{(\nu, \xi')}^{(\nu, \nu)}) + (DD_3 + DD_{12} + DD_{15} + DD_{18} + DD_{21}) N_{(\nu, \xi')}^{(\nu, \nu)}
 \end{aligned}
 \tag{II-160}$$

and $A_{(\nu, q')}^{(2\nu, q)}$, $B_{(\nu, q')}^{(2\nu, q)}$ etc. are the constants that resulted from the expansion (in Dini Series) of the ψ dependent portions of the inhomogeneous parts of the third order equations. The definitions of these constants are available in Appendix C. In the following sections the definitions of the quantities which appear in Equations (II-157) through (II-160) will be given. The reader who is not interested in following the exact details of the analysis may proceed directly to page 83.

$$AA_1 = \frac{1}{4} \bar{C}^2 \left(3i\omega^{(1)} R_{(2m, 2\nu, \xi)}^{(2)} R'' + \bar{f}^{-2} \left((R_{(2m, 2\nu, \xi)}^{(2)})_{\phi} (R'' - U''') + (R''')_{\phi} (R_{(2m, 2\nu, \xi)}^{(2)} - U_{(2m, 2\nu, \xi)}^{(2)}) \right) \right)$$

$$AA_2 = \frac{1}{16} \bar{C}^2 \left(-(i\omega^{(1)} R''')^2 + \bar{f}^{-2} R''^2 (R''')_{\phi} \right) + \bar{f}^{-2} U''' R'' (R''')_{\phi}$$

$$AA_3 = -\frac{1}{2} \bar{C}^2 \bar{f} \bar{q} \left(\Phi_{(2m, 2\nu, \xi)}^{(2)} R'' + \Phi''' R_{(2m, 2\nu, \xi)}^{(2)} \right)$$

$$AA_4 = \frac{1}{8} \bar{C}^2 \bar{f} \bar{q} \Phi''' R''^2$$

$$AA_5 = -\frac{1}{2} AA_3$$

$$AA_6 = -\frac{1}{4} AA_4$$

$$AA_7 = -\frac{1}{4} \bar{q}^2 \bar{c}^2 \left(R_{(2m, 2\nu, \delta)}^{(2)} (R''')_{\varphi} + R'' (R_{(2m, 2\nu, \delta)}^{(2)})_{\varphi} \right)$$

$$AA_8 = -\frac{1}{16} \bar{q}^2 \bar{c}^2 \left(\left(\frac{1}{2} \delta - \frac{1}{6} \delta^2 - \frac{1}{3} \right) \left(3 \frac{\bar{q}^2}{\bar{c}^2} R''^2 (R''')_{\varphi} + \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} R''^3 - \frac{\bar{q}^2}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} R''^3 \right) \right. \\ \left. + \frac{1}{3} (\delta - 2) \left(-\frac{1}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} R''^3 + \frac{3}{\bar{c}^2} R''^2 (R''')_{\varphi} \right) \right)$$

$$AA_9 = -(\delta - 1) AA_7$$

$$AA_{10} = -\frac{1}{16} \bar{q}^2 \bar{c}^2 \left(\frac{5}{2} \delta - \frac{3}{2} \delta^2 - 1 \right) R''^2 (R''')_{\varphi}$$

$$AA_{11} = \frac{3}{4} i \omega^{(n)} (\delta - 1) \bar{c}^2 R'' R_{(2m, 2\nu, \delta)}^{(2)}$$

$$AA_{12} = -\frac{1}{16} i \omega^{(n)} \bar{c}^2 \left(\frac{5}{2} \delta - \frac{3}{2} \delta^2 - 1 \right) R''^3$$

$$AA_{13} = -\frac{1}{(\delta - 1)} AA_{11}$$

$$AA_{14} = -\frac{3}{16} i \omega^{(n)} \left(\left(\frac{1}{2} \delta - \frac{1}{6} \delta^2 - \frac{1}{3} \right) \bar{q}^2 + \frac{1}{3} (\delta - 2) \right) R''^3$$

$$AA_{15} = -\frac{1}{4} \bar{c}^2 \bar{q}^2 \left(-\frac{\bar{q}^2}{\bar{c}^2} \left(U_{(2m, 2\nu, \delta)}^{(2)} (U''')_{\varphi} + (U_{(2m, 2\nu, \delta)}^{(2)})_{\varphi} U'' \right) \right. \\ \left. + \frac{\bar{q}^2}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} U_{(2m, 2\nu, \delta)}^{(2)} U'' - \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} U_{(2m, 2\nu, \delta)}^{(2)} U'' \right)$$

$$AA_{16} = -\frac{1}{4} \bar{c}^2 \bar{q}^2 \left(-\frac{1}{\bar{c}^2} 2 \bar{p} \bar{q} \left((\Phi'')_{\varphi} \Phi_{(2m, 2\nu, \delta)}^{(2)} + \Phi'' (\Phi_{(2m, 2\nu, \delta)}^{(2)})_{\varphi} \right) \right. \\ \left. + \left(\frac{1}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} \bar{p} \bar{q} - \frac{d\bar{p} \bar{q}}{d\varphi} \frac{1}{\bar{c}^2} \right) \Phi'' \Phi_{(2m, 2\nu, \delta)}^{(2)} \right)$$

$$AA_{17} = -\frac{1}{2} AA_{16}$$

$$AA_{18} = \frac{3}{4} i\omega^{(0)} \bar{q}^2 (\Phi^{(1)})_{\varphi} (\Phi_{(2m, 2\nu, \delta)}^{(2)})_{\varphi}$$

$$AA_{19} = \frac{3}{2} i\omega^{(0)} \bar{p} \bar{q} \Phi^{(1)} \Phi_{(2m, 2\nu, \delta)}^{(2)}$$

$$AA_{20} = -\frac{1}{2} AA_{19}$$

(II-161)

$$\begin{aligned} BB_1 = & \frac{1}{4} \bar{C}^2 \left(R^{(1)*} (i\omega^{(0)} R_{(2m, 2\nu, \delta)}^{(2)} + \bar{q}^2 (R_{(2m, 2\nu, \delta)}^{(2)})_{\varphi}) + \bar{q}^2 R_{(2m, 2\nu, \delta)}^{(2)} (R^{(1)*})_{\varphi} \right. \\ & - \bar{q}^2 (U_{(2m, 2\nu, \delta)}^{(2)} (R^{(1)*})_{\varphi} + U^{(1)*} (R_{(2m, 2\nu, \delta)}^{(2)})_{\varphi}) + \bar{q}^2 R^{(1)} (R_{(0, 2\nu, \delta)}^{(2)} + R_{(0, 2\nu, \delta)}^{(2)*}) + (R_{(0, 2\nu, \delta)}^{(2)} \\ & \left. + R_{(0, 2\nu, \delta)}^{(2)*}) (i\omega^{(0)} R^{(1)} + \bar{q}^2 (R^{(1)})_{\varphi}) - \bar{q}^2 (U^{(1)} (R_{(0, 2\nu, \delta)}^{(2)} + R_{(0, 2\nu, \delta)}^{(2)*}) + (R^{(1)})_{\varphi} (U_{(0, 2\nu, \delta)}^{(2)} + U_{(0, 2\nu, \delta)}^{(2)*})) \right) \end{aligned}$$

$$\begin{aligned} BB_2 = & \frac{1}{16} \bar{C}^2 \left(-i\omega^{(0)} R^{(1)2} R^{(1)*} - \bar{q}^2 (R^{(1)2} (R^{(1)*})_{\varphi} + 2R^{(1)} R^{(1)*} (R^{(1)})_{\varphi}) + \bar{q}^2 (U^{(1)} R^{(1)} (R^{(1)*})_{\varphi} \right. \\ & \left. + U^{(1)} R^{(1)*} (R^{(1)})_{\varphi} + U^{(1)*} R^{(1)} (R^{(1)})_{\varphi}) \right) \end{aligned}$$

$$\begin{aligned} BB_3 = & -\frac{1}{2} \bar{C}^2 \bar{p} \bar{q} \left(\Phi_{(2m, 2\nu, \delta)}^{(2)} R^{(1)*} + R_{(2m, 2\nu, \delta)}^{(2)} \Phi^{(1)*} + R^{(1)} (\Phi_{(0, 2\nu, \delta)}^{(2)} + \Phi_{(0, 2\nu, \delta)}^{(2)*}) \right. \\ & \left. + \Phi^{(1)} (R_{(0, 2\nu, \delta)}^{(2)} + R_{(0, 2\nu, \delta)}^{(2)*}) \right) \end{aligned}$$

$$BB_4 = \frac{1}{8} \bar{C}^2 \bar{p} \bar{q} (2\Phi^{(1)} R^{(1)} R^{(1)*} + \Phi^{(1)*} R^{(1)2})$$

$$BB_5 = -\frac{1}{2} BB_3$$

$$BB_6 = -\frac{1}{4} BB_4$$

$$\begin{aligned} \text{BB}_7 = & -\frac{1}{4} \bar{q}^2 \bar{C}^2 \left(R_{(2m, 2\mu, q)}^{(2)} (R^{(1)*})_\varphi + R^{(1)*} (R_{(2m, 2\mu, q)}^{(2)})_\varphi \right. \\ & \left. + (R^{(1)})_\varphi (R_{(0, 2\mu, q)}^{(2)} + R_{(0, 2\mu, q)}^{(2)*}) + R^{(1)} (R_{(0, 2\mu, q)}^{(2)} + R_{(0, 2\mu, q)}^{(2)*})_\varphi \right) \end{aligned}$$

$$\begin{aligned} \text{BB}_8 = & -\frac{1}{16} \bar{q}^2 \bar{C}^2 \left(\left(\frac{1}{2} \gamma - \frac{1}{6} \gamma^2 - \frac{1}{3} \right) \left(3 \frac{\bar{q}^2}{\bar{C}^2} (R^{(1)2} (R^{(1)*})_\varphi + 2 R^{(1)} R^{(1)*} (R^{(1)})_\varphi \right) \right. \\ & \left. + \left(\frac{1}{\bar{C}^2} \frac{d\bar{q}^2}{d\varphi} - \frac{\bar{q}^2}{\bar{C}^4} \frac{d\bar{C}^2}{d\varphi} \right) 3 R^{(1)} R^{(1)*} \right) + \frac{1}{3} (\gamma - 2) \left(-\frac{1}{\bar{C}^4} \frac{d\bar{C}^2}{d\varphi} 3 R^{(1)2} R^{(1)*} \right. \\ & \left. + \frac{1}{\bar{C}^2} 3 (2 R^{(1)} R^{(1)*} (R^{(1)})_\varphi + R^{(1)2} (R^{(1)*})_\varphi) \right) \end{aligned}$$

$$\text{BB}_9 = -(\gamma - 1) \text{BB}_7$$

$$\text{BB}_{10} = -\frac{1}{16} \bar{q}^2 \bar{C}^2 \left(\left(\frac{5}{2} \gamma - \frac{3}{2} \gamma^2 - 1 \right) (2 R^{(1)} R^{(1)*} (R^{(1)})_\varphi + R^{(1)2} (R^{(1)*})_\varphi) \right)$$

$$\text{BB}_{11} = \frac{\gamma - 1}{4} i \omega^{(1)} \bar{C}^2 \left(R_{(2m, 2\mu, q)}^{(2)} R^{(1)*} + R^{(1)} (R_{(0, 2\mu, q)}^{(2)} + R_{(0, 2\mu, q)}^{(2)*}) \right)$$

$$\text{BB}_{12} = -\frac{i \omega^{(1)}}{16} \bar{C}^2 \left(\left(\frac{5}{2} \gamma - \frac{3}{2} \gamma^2 - 1 \right) R^{(1)2} R^{(1)*} \right)$$

$$\text{BB}_{13} = -\frac{1}{(\gamma - 1)} \text{BB}_{11}$$

$$\text{BB}_{14} = -\frac{i \omega^{(1)}}{16} \left(\left(\frac{\gamma}{2} - \frac{1}{6} \gamma^2 - \frac{1}{3} \right) \bar{q}^2 + \frac{1}{3} (\gamma - 2) \right) 3 R^{(1)2} R^{(1)*}$$

$$\begin{aligned} \text{BB}_{15} = & -\frac{1}{4} \bar{c}^2 \bar{g}^2 \left(-\frac{\bar{g}^2}{\bar{c}^2} \left((\Phi^{(1)})^* \right)_\varphi \varphi (\Phi^{(2)}_{(2m, 2\nu, \bar{g})})_\varphi + (\Phi^{(1)})^* (\Phi^{(2)}_{(2m, 2\nu, \bar{g})})_\varphi + (\Phi^{(1)})_\varphi (\Phi^{(2)}_{(0, 2\nu, \bar{g})}) \right. \\ & + \left. \Phi^{(2)*}_{(0, 2\nu, \bar{g})} \right)_\varphi + (\Phi^{(1)})_\varphi (\Phi^{(2)}_{(0, 2\nu, \bar{g})} + \Phi^{(2)*}_{(0, 2\nu, \bar{g})})_\varphi \varphi + \left(\frac{\bar{g}^2}{\bar{c}^2} \frac{d\bar{c}^2}{d\varphi} - \frac{1}{\bar{c}^2} \frac{d\bar{g}^2}{d\varphi} \right) (\Phi^{(1)})^* (\Phi^{(2)}_{(2m, 2\nu, \bar{g})})_\varphi \\ & + (\Phi^{(1)})_\varphi (\Phi^{(2)}_{(0, 2\nu, \bar{g})} + \Phi^{(2)*}_{(0, 2\nu, \bar{g})})_\varphi \varphi \end{aligned}$$

$$\begin{aligned} \text{BB}_{16} = & -\frac{1}{4} \bar{c}^2 \bar{g}^2 \left(-\frac{2\bar{p}\bar{g}}{\bar{c}^2} \left((\Phi^{(1)})^* \right)_\varphi \Phi^{(2)}_{(2m, 2\nu, \bar{g})} + \Phi^{(1)*} (\Phi^{(2)}_{(2m, 2\nu, \bar{g})})_\varphi + (\Phi^{(1)})_\varphi (\Phi^{(2)}_{(0, 2\nu, \bar{g})}) \right. \\ & + \left. \Phi^{(2)*}_{(0, 2\nu, \bar{g})} \right)_\varphi + \Phi^{(1)} (\Phi^{(2)}_{(0, 2\nu, \bar{g})} + \Phi^{(2)*}_{(0, 2\nu, \bar{g})})_\varphi + \left(\frac{\bar{g}\bar{p}}{\bar{c}^2} \frac{d\bar{c}^2}{d\varphi} - \frac{1}{\bar{c}^2} \frac{d(\bar{p}\bar{g})}{d\varphi} \right) (\Phi^{(1)*} \Phi^{(2)}_{(2m, 2\nu, \bar{g})} \\ & + \Phi^{(1)} (\Phi^{(2)}_{(0, 2\nu, \bar{g})} + \Phi^{(2)*}_{(0, 2\nu, \bar{g})})_\varphi \end{aligned}$$

$$\text{BB}_{17} = -\frac{1}{2} \text{BB}_{16}$$

$$\text{BB}_{18} = \frac{1}{4} i\omega^{(0)} \bar{g}^2 \left(U^{(1)*} U^{(2)}_{(2m, 2\nu, \bar{g})} + U^{(1)} (U^{(2)}_{(0, 2\nu, \bar{g})} + U^{(2)*}_{(0, 2\nu, \bar{g})}) \right)$$

$$\text{BB}_{19} = \frac{1}{2} i\omega^{(0)} \bar{p} \bar{g} \left(\Phi^{(1)*} \Phi^{(2)}_{(2m, 2\nu, \bar{g})} + \Phi^{(1)} (\Phi^{(2)}_{(0, 2\nu, \bar{g})} + \Phi^{(2)*}_{(0, 2\nu, \bar{g})}) \right)$$

$$\text{BB}_{20} = -\frac{1}{2} \text{BB}_{19}$$

(II-162)

$$\begin{aligned} \text{CC}_1 = & \frac{1}{4} \bar{c}^2 \left(R^{(1)} (2i\omega^{(0)} R^{(2)}_{(2m, 2\nu, \bar{g})} + \bar{g}^2 (R^{(2)}_{(2m, 2\nu, \bar{g})})_\varphi) + R^{(2)}_{(2m, 2\nu, \bar{g})} (i\omega^{(0)} R^{(1)} \right. \\ & + \left. \bar{g}^2 (R^{(1)})_\varphi) - \bar{g}^2 (U^{(1)} (R^{(2)}_{(2m, 2\nu, \bar{g})})_\varphi + U^{(2)}_{(2m, 2\nu, \bar{g})} (R^{(1)})_\varphi) \right) \end{aligned}$$

$$cc_2 = \frac{1}{2} \bar{C}^2 \left(R^{(1)} (2i\omega^{(1)} R_{(2m,0,\xi)}^{(2)} + \bar{q}^2 (R_{(2m,0,\xi)}^{(2)})_\varphi) + R_{(2m,0,\xi)}^{(2)} (i\omega^{(1)} R^{(1)} + \bar{q}^2 (R^{(1)})_\varphi) \right. \\ \left. - \bar{q}^2 (U^{(1)} (R_{(2m,0,\xi)}^{(2)})_\varphi + U_{(2m,0,\xi)}^{(2)} (R^{(1)})_\varphi) \right)$$

$$cc_3 = 3(AA_2)$$

$$cc_4 = AA_3$$

$$cc_5 = -\bar{q} \bar{C}^2 \left(\Phi_{(2m,0,\xi)}^{(2)} R^{(1)} + R_{(2m,0,\xi)}^{(2)} \Phi^{(1)} \right)$$

$$cc_6 = 3(AA_4)$$

$$cc_7 = -AA_5$$

$$cc_8 = 0$$

$$cc_9 = -AA_6$$

$$cc_{10} = AA_7$$

$$cc_{11} = -\frac{1}{2} \bar{q}^2 \bar{C}^2 \left((R^{(1)})_\varphi R_{(2m,0,\xi)}^{(2)} + R^{(1)} (R_{(2m,0,\xi)}^{(2)})_\varphi \right)$$

$$cc_{12} = 3(AA_8)$$

$$cc_{13} = AA_9$$

$$CC_{14} = -(\gamma-1) CC_{11}$$

$$CC_{15} = 3(AA_{10})$$

$$CC_{16} = AA_{11}$$

$$CC_{17} = \frac{3}{2}(\gamma-1)i\omega^{\omega}\bar{c}^2 R^{(1)} R_{(2m,0,q)}^{(2)}$$

$$CC_{18} = 3(AA_{12})$$

$$CC_{19} = AA_{13}$$

$$CC_{20} = -\frac{1}{(\gamma-1)} CC_{17}$$

$$CC_{21} = 3(AA_{14})$$

$$CC_{22} = AA_{15}$$

$$CC_{23} = -\frac{1}{2}\bar{q}^2\bar{c}^2\left(-\frac{\bar{q}^2}{\bar{c}^2}\left((U^{(1)})_{\varphi}U_{(2m,0,q)}^{(2)}+U^{(1)'}(U_{(2m,0,q)}^{(2)})_{\varphi}\right)+\left(\frac{\bar{q}^2}{\bar{c}^4}\frac{d\bar{c}^2}{d\varphi}-\frac{1}{\bar{c}^2}\frac{d\bar{q}^2}{d\varphi}\right)U^{(1)}U_{(2m,0,q)}^{(2)}\right)$$

$$CC_{24} = AA_{16}$$

$$CC_{25} = -\frac{\bar{q}^2\bar{c}^2}{2}\left(-\frac{2\bar{p}\bar{q}}{\bar{c}^2}\left((\Phi^{(1)})_{\varphi}\Phi_{(2m,0,q)}^{(2)}+\Phi^{(1)'}(\Phi_{(2m,0,q)}^{(2)})_{\varphi}\right)+\left(\frac{2\bar{p}\bar{q}}{\bar{c}^4}\frac{d\bar{c}^2}{d\varphi}-\frac{1}{\bar{c}^2}\frac{d(\bar{p}\bar{q})}{d\varphi}\right)\Phi^{(1)}\Phi_{(2m,0,q)}^{(2)}\right)$$

$$CC_{26} = \frac{1}{2}(AA_{16})$$

$$CC_{27} = 0$$

$$CC_{28} = AA_{18}$$

$$CC_{29} = \frac{3}{2} i\omega^{(2)} \bar{q}^2 U_{(2m,0,g)}^{(2)} U^{(1)}$$

$$CC_{30} = AA_{19}$$

$$CC_{31} = 3 i\omega^{(2)} \bar{f} \bar{q} \Phi_{(2m,0,g)}^{(2)} \Phi^{(1)}$$

$$CC_{32} = \frac{1}{2} AA_{19}$$

$$CC_{33} = 0$$

$$D_1 = BB_1$$

$$\begin{aligned} DD_2 = & \frac{1}{2} \bar{C}^2 \left(R^{(1)*} \left(i\omega^{(2)} R_{(2m,0,g)}^{(2)} + \bar{q}^2 (R_{(2m,0,g)}^{(2)})_{\varphi} \right) + \bar{q}^2 R_{(2m,0,g)}^{(2)} (R^{(1)*})_{\varphi} \right. \\ & - \bar{q}^2 \left(U^{(1)*} (R_{(2m,0,g)}^{(2)})_{\varphi} + U_{(2m,0,g)}^{(2)} (R^{(1)*})_{\varphi} \right) + \bar{q}^2 R^{(1)} (R_{(0,0,g)}^{(2)} + R_{(0,0,g)}^{(2)*}) \\ & + (R_{(0,0,g)}^{(2)} + R_{(0,0,g)}^{(2)*}) (i\omega^{(2)} R^{(1)} + \bar{q}^2 (R^{(1)})_{\varphi}) - \bar{q}^2 \left(U^{(1)} (R_{(0,0,g)}^{(2)} + R_{(0,0,g)}^{(2)*})_{\varphi} \right. \\ & \left. \left. + (U_{(0,0,g)}^{(2)} + U_{(0,0,g)}^{(2)*}) (R^{(1)})_{\varphi} \right) \right) \end{aligned}$$

$$DD_3 = 3(BB_2)$$

$$DD_4 = BB_3$$

$$DD_5 = -\bar{c}^2 \bar{q} \left(\Phi_{(2m,0,f)}^{(2)} R^{(1)*} + R_{(2m,0,f)}^{(2)} \Phi^{(1)*} + R^{(1)} (\Phi_{(0,0,f)}^{(2)} + \Phi_{(0,0,f)}^{(2)*}) + \Phi^{(1)} (R_{(0,0,f)}^{(2)} + R_{(0,0,f)}^{(2)*}) \right)$$

$$DD_6 = 3(BB_4)$$

$$DD_7 = -BB_5$$

$$DD_8 = 0$$

$$DD_9 = -BB_6$$

$$DD_{10} = BB_7$$

$$DD_{11} = -\frac{1}{2} \bar{q}^2 \bar{c}^2 \left(R_{(2m,0,f)}^{(2)} (R^{(1)*})_\varphi + (R_{(2m,0,f)}^{(2)})_\varphi R^{(1)*} + (R^{(1)})_\varphi (R_{(0,0,f)}^{(2)} + R_{(0,0,f)}^{(2)*}) + R^{(1)} (R_{(0,0,f)}^{(2)} + R_{(0,0,f)}^{(2)*}) \right)$$

$$DD_{12} = 3(BB_8)$$

$$DD_{13} = BB_9$$

$$DD_{14} = -(\gamma-1) DD_{11}$$

$$DD_{15} = 3(BB_{10})$$

$$DD_{16} = BB_{11}$$

$$DD_{17} = \frac{1}{2} i \omega^2 \bar{c}^2 (\gamma-1) (R_{(2m,0,\delta)}^{(2)} R^{(1)*} + R^{(1)} (R_{(0,0,\delta)}^{(2)} + R_{(0,0,\delta)}^{(2)*}))$$

$$DD_{18} = 3(BB_{12})$$

$$DD_{19} = BB_{13}$$

$$DD_{20} = -\frac{1}{(\gamma-1)} DD_{17}$$

$$DD_{21} = 3(BB_{14})$$

$$DD_{22} = BB_{15}$$

$$DD_{23} = -\frac{1}{2} \bar{q}^2 \bar{c}^2 \left(-\frac{\bar{q}^2}{\bar{c}^2} \left((U_{(2m,0,\delta)}^{(2)})_{,\varphi} U^{(1)*} + U_{(2m,0,\delta)}^{(2)} (U^{(1)})_{,\varphi} + U^{(1)} (U_{(0,0,\delta)}^{(2)} + U_{(0,0,\delta)}^{(2)*})_{,\varphi} \right. \right. \\ \left. \left. + (U^{(1)})_{,\varphi} (U_{(0,0,\delta)}^{(2)} + U_{(0,0,\delta)}^{(2)*}) + \left(\frac{\bar{q}^2}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} - \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} \right) (U_{(2m,0,\delta)}^{(2)} U^{(1)*} + U^{(1)} (U_{(0,0,\delta)}^{(2)} + U_{(0,0,\delta)}^{(2)*})) \right) \right)$$

$$DD_{24} = BB_{16}$$

$$\begin{aligned}
 DD_{25} = & -\bar{q}^2 \bar{c}^2 \left(-\bar{p} \bar{q} \left((\Phi_{(2m,0,\delta)}^{(2)})_{\varphi} \Phi^{(1)} + (\Phi^{(1)*})_{\varphi} \right. \right. \\
 & + \Phi^{(1)} \left(\Phi_{(0,0,\delta)}^{(1)} + \Phi_{(0,0,\delta)}^{(2)*} \right)_{\varphi} + (\Phi^{(1)})_{\varphi} \left(\Phi_{(0,0,\delta)}^{(2)} + \Phi_{(0,0,\delta)}^{(2)*} \right) \left. \right) \frac{1}{\bar{c}^2} \\
 & + \left(\frac{1}{\bar{c}^4} \bar{p} \bar{q} \frac{d\bar{c}^2}{d\varphi} - \frac{1}{\bar{c}^2} \frac{d(\bar{p}\bar{q})}{d\varphi} \right) \left(\Phi^{(1)*} \Phi_{(2m,0,\delta)}^{(2)} + \Phi^{(1)} \left(\Phi_{(0,0,\delta)}^{(2)} \right. \right. \\
 & \left. \left. + \Phi_{(0,0,\delta)}^{(2)*} \right) \right)
 \end{aligned}$$

$$DD_{26} = \frac{1}{2} DD_{24}$$

$$DD_{27} = 0$$

$$DD_{28} = BB_{18}$$

$$DD_{29} = \frac{1}{2} i\omega^{(m)} \bar{q}^{-1} \left(U_{(2m,0,\delta)}^{(2)} U^{(1)*} + U^{(1)} \left(U_{(0,0,\delta)}^{(2)} + U_{(0,0,\delta)}^{(2)*} \right) \right)$$

$$DD_{30} = BB_{19}$$

$$DD_{31} = i\omega^{(m)} \bar{p} \bar{q} \left(\Phi_{(2m,0,\delta)}^{(2)} \Phi^{(1)*} + \Phi^{(1)} \left(\Phi_{(0,0,\delta)}^{(2)} + \Phi_{(0,0,\delta)}^{(2)*} \right) \right)$$

$$DD_{32} = \frac{1}{2} DD_{30}$$

$$DD_{33} = 0$$

The expressions derived so far were applicable to the case of standing-wave motion. In the case of travelling-wave motion the inhomogeneous part of Equation (II-61) can be written in the following form:

$$I_T^{(3)} = \sum_{q'=1}^{\infty} I_{(3m, 2\nu, q')}^{(3)} e^{3i(\nu\theta + m\gamma)} J_{3\nu}(S_{(3\nu, q')}\sqrt{\frac{\psi'}{\psi_w}}) + \sum_{q'=1}^{\infty} I_{(m, \nu, q')}^{(3)} e^{i(\nu\theta + m\gamma)} J_{\nu}(S_{(\nu, q')}\sqrt{\frac{\psi'}{\psi_w}}) \quad (\text{II-165})$$

where

$$I_{(3m, 3\nu, q')}^{(3)} = \sum_{q=1}^{\infty} \left\{ (AA_1 + AA_7 + AA_9 + AA_{11} + AA_{13} + AA_{15} + AA_{18}) 2A_{(3\nu, q')}^{(2\nu, q)} + \frac{2}{\psi_w} \left((AA_3 + AA_{16} + AA_{19}) B_{(3\nu, q')}^{(2\nu, q)} + (AA_5 + AA_{17} + AA_{20}) C_{(3\nu, q')}^{(2\nu, q)} \right) + \frac{4}{\psi_w} (AA_4 D_{(3\nu, q')}^{(\nu, \nu)} + AA_6 M_{(3, q')}^{(\nu, \nu)}) + 4N_{(3\nu, q')}^{(\nu, \nu)} (AA_2 + AA_8 + AA_{10} + AA_{12} + AA_{14}) \right\} \quad (\text{II-166})$$

and

$$I_{(m, \nu, q')}^{(3)} = \sum_{q=0}^{\infty} \left\{ (EE_1 + EE_{10} + EE_{13} + EE_{16} + EE_{19} + EE_{22} + EE_{28}) 2A_{(\nu, q')}^{(2\nu, q)} + (EE_2 + EE_{11} + EE_{14} + EE_{17} + EE_{20} + EE_{23} + EE_{29}) 2A_{(\nu, q')}^{(\nu, q)} + \frac{2}{\psi_w} \left((EE_4 + EE_{24} + EE_{30}) B_{(\nu, q')}^{(2\nu, q)} + (EE_5 + EE_{25} + EE_{31}) B_{(\nu, q')}^{(\nu, q)} + (EE_7 + EE_{26} + EE_{32}) C_{(\nu, q')}^{(2\nu, q)} \right) + \frac{4}{\psi_w} (EE_6 D_{(\nu, q')}^{(\nu, \nu)} + EE_9 M_{(\nu, q')}^{(\nu, \nu)}) \right\}$$

$$+ 4N_{(\nu, \delta)}^{(\nu, \nu)} (EE_3 + EE_{12} + EE_{15} + EE_{18} + EE_{21})$$

(II-167)

the definitions of AA_1 through AA_{20} were given previously in Equation (II-161) and

$$EE_1 = \frac{\bar{c}^2}{4} \left(R^{(1)*} (i\omega^{(1)} R_{(2m, 2\nu, \delta)}^{(2)} + \bar{q}^2 (R_{(2m, 2\nu, \delta)}^{(2)})_{\varphi}) + \bar{q}^2 (R_{(2m, 2\nu, \delta)}^{(2)} (R^{(1)*})_{\varphi} \right. \\ \left. - U_{(2m, 2\nu, \delta)}^{(2)} (R^{(1)*})_{\varphi} - (R_{(2m, 2\nu, \delta)}^{(2)})_{\varphi} U^{(1)*} \right)$$

$$EE_2 = \frac{\bar{c}^2}{4} \left(\bar{q}^2 R^{(1)} (R_{(0, 0, \delta)}^{(2)} + R_{(0, 0, \delta)}^{(2)*})_{\varphi} + (R_{(0, 0, \delta)}^{(2)} + R_{(0, 0, \delta)}^{(2)*}) (i\omega^{(1)} R^{(1)} + \bar{q}^2 (R^{(1)})_{\varphi} \right. \\ \left. - \bar{q}^2 (U^{(1)} (R_{(0, 0, \delta)}^{(2)} + R_{(0, 0, \delta)}^{(2)*})_{\varphi} + (R^{(1)})_{\varphi} (U_{(0, 0, \delta)}^{(2)} + U_{(0, 0, \delta)}^{(2)*})) \right)$$

$$EE_3 = BB_2$$

$$EE_4 = -\frac{\bar{c}^2}{2} \bar{p} \bar{q} \left(\Phi_{(2m, 2\nu, \delta)}^{(2)} R^{(1)*} + R_{(2m, 2\nu, \delta)}^{(2)} \Phi^{(1)*} \right)$$

$$EE_5 = -\frac{\bar{c}^2}{2} \bar{p} \bar{q} \left(R^{(1)} (\Phi_{(0, 0, \delta)}^{(2)} + \Phi_{(0, 0, \delta)}^{(2)*}) + \Phi^{(1)} (R_{(0, 0, \delta)}^{(2)} + R_{(0, 0, \delta)}^{(2)*}) \right)$$

$$EE_6 = EE_4$$

$$EE_7 = -\frac{1}{4} \bar{c}^2 \bar{p} \bar{q} \left(\Phi_{(2, 2, q)}^{(2)} R^{(1)*} + R_{(2, 2, q)}^{(2)} \Phi^{(1)*} \right)$$

$$EE_8 = 0$$

$$EE_9 = -BB_6$$

$$EE_{10} = -\frac{1}{4} \bar{g}^2 \bar{c}^2 \left(R_{(2m, 2n, \delta)}^{(2)} (R^{(1)*})_\varphi + (R_{(2m, 2n, \delta)}^{(2)})_\varphi R^{(1)*} \right)$$

$$EE_{11} = -\frac{1}{4} \bar{g}^2 \bar{c}^2 \left((R^{(1)})_\varphi (R_{(0,0,\delta)}^{(2)} + R_{(0,0,\delta)}^{(2)*}) + R^{(1)} (R_{(0,0,\delta)}^{(2)} + R_{(0,0,\delta)}^{(2)*})_\varphi \right)$$

$$EE_{12} = BB_8$$

$$EE_{13} = -(\delta-1) EE_{10}$$

$$EE_{14} = -(\delta-1) EE_{11}$$

$$EE_{15} = BB_{10}$$

$$EE_{16} = \frac{\bar{c}^2(\delta-1)}{4} i\omega^{(0)} R_{(2m, 2n, \delta)}^{(2)} R^{(1)*}$$

$$EE_{17} = \frac{\bar{c}^2(\delta-1)}{4} i\omega^{(0)} R^{(1)} (R_{(0,0,\delta)}^{(2)} + R_{(0,0,\delta)}^{(2)*})$$

$$EE_{18} = BB_{12}$$

$$EE_{19} = -\frac{1}{(\delta-1)} EE_{16}$$

$$EE_{20} = -\frac{1}{(\delta-1)} EE_{17}$$

$$EE_{21} = BB_{14}$$

$$EE_{22} = -\frac{\bar{q}^2 \bar{c}^2}{4} \left(-\frac{\bar{q}^2}{\bar{c}^2} \left(U_{(\lambda m, \lambda \nu, \delta)}^{(2)} (U^{(1)*})_{\varphi} + (U_{(\lambda m, \lambda \nu, \delta)}^{(2)})_{\varphi} U^{(1)*} \right) + \left(\frac{\bar{q}^2}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} - \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} \right) U_{(\lambda m, \lambda \nu, \delta)}^{(2)} U^{(1)*} \right)$$

$$EE_{23} = -\frac{\bar{q}^2 \bar{c}^2}{4} \left(-\frac{\bar{q}^2}{\bar{c}^2} \left((U^{(1)})_{\varphi} (U_{(0,0,\delta)}^{(2)} + U_{(0,0,\delta)}^{(2)*}) + U^{(1)} (U_{(0,0,\delta)}^{(2)} + U_{(0,0,\delta)}^{(2)*})_{\varphi} \right) + \left(\frac{\bar{q}^2}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} - \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} \right) U^{(1)} (U_{(0,0,\delta)}^{(2)} + U_{(0,0,\delta)}^{(2)*}) \right)$$

$$EE_{24} = -\frac{\bar{q}^2 \bar{c}^2}{2} \left(-\frac{\bar{p}\bar{q}}{\bar{c}^2} \left(\Phi_{(\lambda m, \lambda \nu, \delta)}^{(2)} (\Phi^{(1)*})_{\varphi} + (\Phi_{(\lambda m, \lambda \nu, \delta)}^{(2)})_{\varphi} \Phi^{(1)*} \right) + \left(\frac{\bar{p}\bar{q}}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} - \frac{1}{\bar{c}^2} \frac{d(\bar{p}\bar{q})}{d\varphi} \right) \Phi_{(\lambda m, \lambda \nu, \delta)}^{(2)} \Phi^{(1)*} \right)$$

$$EE_{25} = -\frac{\bar{q}^2 \bar{c}^2}{2} \left(-\frac{\bar{p}\bar{q}}{\bar{c}^2} \left((\Phi^{(1)})_{\varphi} (\Phi_{(0,0,\delta)}^{(2)} + \Phi_{(0,0,\delta)}^{(2)*}) + \Phi^{(1)} (\Phi_{(0,0,\delta)}^{(2)} + \Phi_{(0,0,\delta)}^{(2)*})_{\varphi} \right) + \left(\frac{\bar{p}\bar{q}}{\bar{c}^4} \frac{d\bar{c}^2}{d\varphi} - \frac{1}{\bar{c}^2} \frac{d(\bar{p}\bar{q})}{d\varphi} \right) \Phi^{(1)} (\Phi_{(0,0,\delta)}^{(2)} + \Phi_{(0,0,\delta)}^{(2)*}) \right)$$

$$EE_{26} = \frac{1}{2} EE_{24}$$

$$EE_{27} = 0$$

$$EE_{28} = \frac{1}{4} i\omega^{(1)} \bar{q}^2 U_{(\lambda m, \lambda \nu, \delta)}^{(2)} U^{(1)*}$$

$$EE_{29} = \frac{1}{4} i\omega^{(1)} \bar{q}^2 U^{(1)} (U_{(0,0,\delta)}^{(2)} + U_{(0,0,\delta)}^{(2)*})$$

$$EE_{30} = \frac{1}{2} i\omega^{(1)} \bar{p}\bar{q} \Phi_{(\lambda m, \lambda \nu, \delta)}^{(2)} \Phi^{(1)*}$$

$$EE_{31} = \frac{1}{2} i \omega'' \bar{f} \bar{q} \Phi'' (\Phi_{(\omega, \theta, \xi)}^{(3)} + \Phi_{(\omega, \theta, \xi)}^{(3)*})$$

$$EE_{32} = \frac{1}{2} EE_{30}$$

(II-168)

To obtain the solutions of the third order equations for both the standing and travelling-wave motion, the dependent variables which appear in the problem are assumed to have eigenfunction expansions similar to either the one given in Equation (II-156) (for $I_s^{(3)}$) in the case of standing-wave motion or Equation (II-165) (for $I_r^{(3)}$) in the case of travelling-wave motion. Following the same procedure as in the second order analysis, the series expansions of the dependent variables as well as the inhomogeneous parts of these equations are then substituted into Equations (II-45) through (II-50).[#] Separating variables in the resulting expressions and grouping together the coefficients of the various eigenfunctions result in an infinite number of systems of ordinary differential equations whose solutions describe the third order motion. The solutions of each of these systems, which can be obtained independently of the other systems, describes the axial dependence of a particular set of coefficients, of a given eigenfunction, which appear in the series expansion of the third order unknowns. Since the derivation and solution of the third order equations are, in principle, identical to the derivation and solution of the second order equation, they will not be given here.

Some Modifications of the Higher Order Solutions

In this section the possibility of the appearance of an eigenfunction with the same transverse and time dependence as the first order solution in the series expansion of the second and third order solutions will be investigated. The presence of $I_{(m, \nu, h)}^{(3)}$ in the eigenfunction expansion of $I^{(3)}$ (see Equation II-156) has resulted in the appearance of

[#] Although not given here explicitly, the expansion of the inhomogeneous parts of these equations can be obtained by using the same procedures as used in the expansion of $I_s^{(3)}$ or $I_r^{(3)}$.

such a component in the third order solution. For convenience in the analysis it will be assumed that the expansion of $I^{(2)}$ contains a similar component (i.e., $I_{(m,\nu,h)}^{(2)}$) which at the conclusion of this study will be taken to be identically zero. Examination of Equations (II-36) through (II-41) reveals that the inhomogeneous parts of these equations contain additional components (which are inside the summation sign) which have not yet been considered in the analysis and which have the same transverse and time dependence as $I_{(m,\nu,h)}^{(2)}$ and $I_{(m,\nu,h)}^{(3)}$. Because of the importance of the solutions (which result from the presence of these expressions in the inhomogeneous part of the higher order equations) in the determination of the stability limits of finite amplitude waves inside combustion chambers of liquid propellant rocket engines, their analysis is given a special consideration. Repeating the analysis of the previous sections (which provided us with the solutions of the second and third order equations) and including in it the terms appearing inside the summation signs present in the inhomogeneous parts of Equations (II-37) through (II-41) and use of the relation $\tilde{\eta}_{00}'' = \tilde{S}_{00}''$ yield the following partial differential Equation for $F_{(m,\nu,q)}^{(j)}$, ($j = 2, 3$):

$$\begin{aligned} \mathcal{F}_{(m,\nu,h)}(F_{(m,\nu,h)}^{(j)}) = & -i\omega^{(k)} \left((S'' + \mathcal{K}'') - \bar{q}^2 \left(\frac{H''}{\bar{c}^2} \right)_\varphi \right) \bar{c}^2 \\ & - im\omega^{(m)} H'' - 2\bar{f}\bar{q}\bar{c}^2 f_0^{(m)} \left(\left(\frac{\partial}{\partial \psi} \left(\psi \frac{\partial}{\partial \psi} \right) + \frac{1}{4\psi} \frac{\partial^2}{\partial \theta^2} \right) \left(\int_0^\varphi \frac{\tilde{\eta}'' - H''}{\bar{q}^2 f_0^{(m)}} \partial \varphi' \right) \right) \\ & + I_{(m,\nu,h)}^{(j)} \mathcal{R}_{\nu}^{(j)} \mathcal{P}_{(m,h)}^{(j)} \end{aligned} \quad (\text{II-169})$$

where $\mathcal{F}_{(m,\nu,h)}(F_{(m,\nu,h)}^{(j)})$ represents the left-hand-side of Equation (II-61)

$$H'' = \int_0^\varphi \frac{1}{2} \frac{d\bar{q}^2}{d\varphi} \bar{S}_{(m,\nu,h)}^{(j)} \partial \varphi' + F'' \quad (\text{II-170})$$

and $\bar{S}_{(m,\nu,h)}^{(j)}$ is the additional component of $S^{(j)}$ that resulted from the inclusion of the terms inside the summation sign in the solution of Equation (II-40).

That is,

$$\bar{S}_{(m,\nu,h)}^{(j)} = f_0^{(m)} \int_0^\varphi \frac{S''}{\bar{q}^2 f_0^{(m)}} \partial \varphi' = S''(0, \psi, \theta) f_0^{(m)} \int_0^\varphi \frac{d\varphi'}{\bar{q}^2}$$

In Equation (II-169) $k = 1$ when $j = 2$ and $k = 2$ when $j = 3$. The last statement holds because, as will be shown later, ω'' is identically zero. By use of the method of separation of variables Equation (II-169) can be readily reduced from a partial differential equation into an ordinary differential equation whose solution will be discussed in a later section.

When the flow under consideration is irrotational the following results are available from previous analysis:

$$\begin{aligned} S^{(1)} &= \bar{S}_{(m,\nu,h)}^{(j)} = 0 \quad \text{for all } \varphi \\ \bar{H}^{(1)} &= F^{(1)} \quad \text{for all } \varphi \\ \tilde{\eta}^{(1)} &= F^{(1)} \quad \text{for all } \varphi \end{aligned} \tag{II-171}$$

Substitution of these relations into Equation (II-169) and use of Equations (II-46a) and (II-50) for $j = 1$ give:

$$\begin{aligned} \mathcal{F}_{(m,\nu,h)}(F_{(m,\nu,h)}^{(j)}) &= -i\omega^{(k)} \left(\bar{C}^2 \mathcal{K}'' - \bar{C}^2 \bar{q}^2 \left(\frac{F''}{\bar{C}^2} \right)_\varphi - im\omega''' F'' \right) + I_{(m,\nu,h)}^{(j)} \otimes_{\nu,(\theta)} \bar{\Psi}_{(\nu,h)}^{(\psi)} \\ &= -i\omega^{(k)} \left(2\bar{C}^2 \mathcal{K}'' + \frac{\bar{q}^2}{\bar{C}^2} \frac{d\bar{C}^2}{d\varphi} F'' \right) + I_{(m,\nu,h)}^{(j)} \otimes_{\nu,(\theta)} \bar{\Psi}_{(\nu,h)}^{(\psi)} \end{aligned}$$

When the assumed form of $F_{(m,\nu,h)}^{(j)}$ (i.e. $F_{(m,\nu,h)}^{(j)} = \Phi_{(m,\nu,h)}^{(j)} \Theta_{(\theta)}^{(j)} \Psi_{(\psi)}^{(j)}$)

and the known solutions of $\chi^{(1)}$ and $F^{(1)}$ are substituted into Equation (II-173) the latter is reduced after separation of variables to the following ordinary differential equation:

$$\mathcal{L}_{(m,\nu,h)}(\Phi_{(m,\nu,h)}^{(j)}) = -i\omega^{(k)}(2\bar{c}^2 R'' + \frac{\bar{q}^2}{\bar{c}^2} \frac{d\bar{c}^2}{d\varphi} \Phi'') + I_{(m,\nu,h)}^{(j)} \quad (\text{II-173})$$

where $\mathcal{L}_{(m,\nu,h)}(\Phi_{(m,\nu,h)}^{(j)})$ is defined in Equation (II-93).

In conclusion we see that each one of the higher order solutions contains one component which is proportional to the eigenfunction of the first order solution. Combining all the differential equations which describe the behavior of these components, in the case of irrotational[#] flow, yields the following result:

$$\begin{aligned} \mathcal{L}_{(m,\nu,h)}(\epsilon \Phi'' + \epsilon^2 \Phi_{(m,\nu,h)}^{(2)} + \epsilon^3 \Phi_{(m,\nu,h)}^{(3)}) \\ = \epsilon^2 \left(-i\omega^{(2)}(2\bar{c}^2 R'' + \frac{\bar{q}^2}{\bar{c}^2} \frac{d\bar{c}^2}{d\varphi} \Phi'') + I_{(m,\nu,h)}^{(2)} \right) \\ + \epsilon^3 \left(-i\omega^{(3)}(2\bar{c}^2 R'' + \frac{\bar{q}^2}{\bar{c}^2} \frac{d\bar{c}^2}{d\varphi} \Phi'') + I_{(m,\nu,h)}^{(3)} \right) \end{aligned} \quad (\text{II-174})^{##}$$

[#] This case is given special consideration since it will be considered in detail in the analysis of Chapter IV.

^{##} Since $\omega^{(1)}$ will be shown to be identically zero, terms proportional to it have not been included in the third order portion of the inhomogeneous part of this equation.

The solution of Equation (II-174), which will be discussed in Chapter IV, plays an important role in the determination of the relationship between the eigenvalue perturbations. The latter will be used in the determination of the stability criterion of finite amplitude irrotational pressure waves.

Derivation of the Admittance Relations

In the previous sections the solution of the equations which describe the motion of finite-amplitude waves inside the converging section of a nozzle, operating in the supercritical range, was discussed in detail. In this analysis the dependent variables were expanded in powers of ϵ and the solution of the equations which describe the behavior of coefficients of these series were obtained in consecutive order. While first order equations could be solved by a straightforward application of the method of separation of variables, then in the solutions of the higher-order equations this method could be applied only after the dependent variables had been expanded in terms of the appropriate eigenfunctions. In both cases, application of the method of separation of variables led to the derivation of the following ordinary differential equation for $\Phi_{(km, n, g)}^{(j)}$:

$$\begin{aligned} & \bar{q}^2 (\bar{c}^2 - \bar{q}^2) \frac{d^2}{d\varphi^2} \Phi_{(km, n, g)}^{(j)} - \bar{q}^2 \left(2ikm\omega^{(0)} + \frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} \right) \frac{d}{d\varphi} \Phi_{(km, n, g)}^{(j)} \\ & + \left(k^2 m^2 \omega^{(0)2} - \frac{1}{\bar{c}^2} \bar{q}^2 \frac{\delta-1}{2} ikm\omega^{(0)} \frac{d\bar{q}^2}{d\varphi} - \frac{S_{(m, g)}^2}{2\psi_w} \bar{\rho} \bar{q} \bar{c}^2 \right) \Phi_{(km, n, g)}^{(j)} = \\ & \sigma_{(km, n, g)}^{(j)} I_{e(km, n, g)}^{(j)} + C_{l(km, n, g)}^{(j)} I_{v(km, n, g)}^{(j)} + I_{N(km, n, g)}^{(j)} \end{aligned}$$

(II-175)

where $\sigma_{(km,n\nu,q)}^{(j)}$ and $C_{1(km,n\nu,q)}^{(j)}$ represent the components of the entropy and vorticity at the nozzle throat respectively. The quantities $I_{e(km,n\nu,q)}^{(j)}$, $I_{v(km,n\nu,q)}^{(j)}$ and $I_{N(km,n\nu,q)}^{(j)}$ which represent in the given order the effects of entropy, vorticity and nonlinearity are assumed to be known. In its present form Equation (II-175) is quite general and can represent the behavior of the first order solution or any of the coefficients (of a given eigenfunction) of the higher order expansions. In first order analysis $I_{N(km,n\nu,q)}^{(1)} = 0$. When the flow under consideration is homentropic then $\sigma_{(km,n\nu,q)}^{(j)} = 0$ and if in addition the flow is irrotational then $C_{1(km,n\nu,q)}^{(j)} = 0$. From the analysis of the previous sections it can be shown that once Equation (II-175) has been solved and $\Phi_{(km,n\nu,q)}^{(j)}$ is known the solution for the remainder of the unknowns, which have the same subscripts and superscript as $\Phi_{(km,n\nu,q)}^{(j)}$, can be easily obtained.

For convenience in later analysis all subscripts and superscripts will be omitted from the following analysis and Equation (II-175) will be rewritten in the following form:

$$\mathcal{L}(\Phi) = \sigma I_e + C_1 I_v + I_N \quad (\text{II-175a})$$

Equation (II-175) is a second order, linear ordinary differential Equation and its general solution is a combination of the homogeneous solution that satisfies the homogeneous part of Equation (II-175a) i.e.,

$$\mathcal{L}(\Phi_h) = 0 \quad (\text{II-176})$$

and the particular solution that satisfies Equation (II-175a). If these solutions are known, the general solution of Equation (II-175a) can be written in the following form:

$$\Phi = C_1 \Phi_v + \sigma \Phi_e + \Phi_N + C_2 \Phi_h + C_3 \bar{\Phi}_h \quad (\text{II-177})$$

where Φ_h and $\bar{\Phi}_h$ are two independent solutions of Equation (II-176) and Φ_v , Φ_e and Φ_N are respectively the particular solutions that resulted from the presence of I_v , I_e and I_N in the inhomogeneous part of Equation (II-175a); C_2 and C_3 are arbitrary constants.

Examination of the coefficients of Equation (II-175) shows that the latter has the following singular points:

$$\begin{aligned} \bar{q} &= 0 \\ \bar{q} &= \bar{c} = \bar{c}_{\text{throat}} = \left(\frac{2}{\gamma+1} \right)^{\frac{1}{2}} \\ \bar{q} &= \infty \end{aligned} \quad (\text{II-178})$$

For a supercritical nozzle with a finite area entrance, only the singularity at the throat (where $\varphi = 0$ and $\bar{q} = \bar{c}$) is of interest to us. Assuming that all the singularity of the solution appears in $\bar{\Phi}_h$, then the condition requiring the regularity of the solution at the nozzle throat can be expressed by requiring $C_3 = 0$. Consequently, the proper solution of (II-175a) is:

$$\Phi = C_1 \Phi_v + \sigma \Phi_e + \Phi_N + C_2 \Phi_h \quad (\text{II-179})$$

Use of Equation (II-179) and substitution of Equations (II-84) and (II-85), in the case of first order analysis, or the appropriate eigenfunction expansions[#] of the higher order variables into Equations

[#] See for example Equations (II-118) through (II-123) which represent the assumed form of the second order solutions.

(II-51), (II-46a), and Equations (E-2) through (E-5) of Appendix E results in the following equations after the variables have been separated:

$$\begin{aligned}
 U - \frac{d}{d\varphi} \Phi_N &= C_1 \frac{d}{d\varphi} \Phi_v & + \sigma \frac{d}{d\varphi} \Phi_e \\
 & & + C_2 \frac{d}{d\varphi} \Phi_h \\
 V - \Phi_N + M &= C_1 (\Phi_v + f_0^{(km)}) & + \sigma (\Phi_e - f_2^{(km)}) \\
 & & + C_2 \Phi_h \\
 P + \bar{q}^2 U + kmi\omega^{(w)} \Phi_N - K &= -C_1 kmi\omega^{(w)} \Phi_v & - \sigma kmi\omega^{(w)} (\Phi_e - \frac{f_1^{(km)}}{kmi\omega^{(w)}}) \\
 & & - C_2 kmi\omega^{(w)} \Phi_h \\
 S - f_0^{(km)} \int_0^\phi \frac{D(\varphi')}{\bar{q}^2 f_0^{(km)}} d\varphi' &= 0 & + \sigma f_0^{(km)} \\
 & & + 0
 \end{aligned}$$

(II-180)

where

$$K = \int_0^\varphi \frac{1}{2} \frac{d\bar{q}^2}{d\varphi} f_0^{(km)} \int_0^\phi \frac{D(\varphi'')}{\bar{q}^2 f_0^{(km)}} d\varphi'' d\varphi' + A(\varphi)$$

(II-181)[#]

[#] Note that as given here, K has the same definition as H which is given in Equation (II-118b).

and M is defined in Equation (E-5) of Appendix E. If at any point ϕ , along the subsonic portion of the nozzle, U , $V = W$, P and S are known, then the above system of linear equations can be used for the determination of C_1 , σ and C_2 . From the structure of Equations (II-180) it follows that, since there are four linear equations and three unknowns, only three of the four quantities U , $V = W$, P and S are independent of one another. In other words, a relation must exist between the four quantities of any such set. This relation is obtained from the compatibility relation of the four linear equations;[#] i.e., from the vanishing of the following determinant

$$\begin{vmatrix}
 U - \frac{d}{d\phi} \Phi_N & \frac{d}{d\phi} \Phi_V & \frac{d}{d\phi} \Phi_e & \frac{d}{d\phi} \Phi_h \\
 V - \Phi_N + M & (\Phi_V + f_0^{(km)}) & (\Phi_e - f_2^{(km)}) & \Phi_h \\
 P + \bar{q}^2 U + km i \omega'' \Phi_N - K & -km i \omega'' \Phi_V & km i \omega'' (\Phi_e - \frac{f_1^{(km)}}{km i \omega''}) & km i \omega'' \Phi_h \\
 S - f_0^{(km)} \int_0^\phi \frac{D(\phi')}{\bar{q}^2 f_0^{(km)}} d\phi' & 0 & f_0^{(km)} & 0
 \end{vmatrix} = 0$$

(II-182)

letting

$$\hat{Z} = f_0^{(km)} \int_0^\phi \frac{D(\phi')}{\bar{q}^2 f_0^{(km)}} d\phi'$$

(II-183)

For discussion of this topic see, for example, Reference 10.

the determinant given in Equation (II-182) can be rewritten in the following form:

$$\begin{array}{c}
 \begin{array}{|ccc|}
 \hline
 U & \frac{d}{d\varphi} \Phi_v & -\left(\frac{f_o^{(km)}}{S}\right)U + \frac{d}{d\varphi} \Phi_e & \frac{d}{d\varphi} \Phi_h \\
 \hline
 V & (\Phi_v + f_o^{(km)}) & -\left(\frac{f_o^{(km)}}{S}\right)V + (\Phi_e - f_z^{(km)}) & \Phi_h \\
 \hline
 P + \bar{q}^2 U & -kmi\omega^{(g)} \Phi_v & -\left(\frac{f_o^{(km)}}{S}\right)(P + \bar{q}^2 U) + (f_i^{(km)} - kmi\omega^{(g)} \Phi_e) & -kmi\omega^{(g)} \Phi_h \\
 \hline
 S & 0 & 0 & 0 \\
 \hline
 \end{array} \\
 \\
 \begin{array}{|ccc|}
 \hline
 \frac{d}{d\varphi} \Phi_N & \frac{d}{d\varphi} \Phi_v & -\left(\frac{f_o^{(km)}}{\bar{Z}}\right)\frac{d}{d\varphi} \Phi_N + \frac{d}{d\varphi} \Phi_e & \frac{d}{d\varphi} \Phi_h \\
 \hline
 \Phi_N - M & (\Phi_v + f_o^{(km)}) & -\left(\frac{f_o^{(km)}}{\bar{Z}}\right)(\Phi_N - M) + (\Phi_e - f_z^{(km)}) & \Phi_h \\
 \hline
 K - kmi\omega^{(g)} \Phi_N & -kmi\omega^{(g)} \Phi_v & -\left(\frac{f_o^{(km)}}{\bar{Z}}\right)(K - kmi\omega^{(g)} \Phi_N) + (f_i^{(km)} - kmi\omega^{(g)} \Phi_e) & -kmi\omega^{(g)} \Phi_h \\
 \hline
 \hat{Z} & 0 & 0 & 0 \\
 \hline
 \end{array}
 \end{array}
 \quad (II-184)$$

Developing the above determinants, dividing by $f_0^{(km)^2} \Phi_h$, using the definitions:

$$\mu = \frac{1}{\Phi_h} \frac{d}{d\varphi} \Phi_h \quad (II-185a)$$

$$\Gamma_x(\varphi) = \frac{1}{\bar{c}^2 f_0^{(km)} \Phi_h} \left(\Phi_x \frac{d}{d\varphi} \Phi_h - \Phi_h \frac{d}{d\varphi} \Phi_x \right) \quad (II-185b)$$

where the subscript x stands for e , v or N we obtain the following general admittance relation

$$\begin{aligned} & U \left(\bar{q}^2 (\bar{c}^2 \Gamma_v - \mu) - kmi\omega^{(0)} \right) + V \left(kmi\omega^{(0)} \bar{c}^2 \Gamma_v \right) \\ & + P (\bar{c}^2 \Gamma_v - \mu) - S \bar{c}^2 \left(\frac{\bar{q}^2 - \bar{q}_{(0)}^2}{2} \Gamma_v + kmi\omega^{(0)} \Gamma_e - f_3^{(km)} \mu \right) \\ & = kmi\omega^{(0)} \bar{c}^2 f_0^{(km)} \Gamma_N - M kmi\omega^{(0)} \Gamma_v + K (\bar{c}^2 \Gamma_v - \mu) \\ & - \hat{Z} \bar{c}^2 \left(\frac{\bar{q}^2 - \bar{q}_{(0)}^2}{2} \Gamma_v + kmi\omega^{(0)} \Gamma_e - f_3^{(km)} \mu \right) \end{aligned} \quad (II-186)$$

where Γ_x , μ , \bar{q} and \bar{c} are assumed to be known. The equations that describe the behavior of these quantities and their solutions will be discussed in the next section.

In addition to the definitions given in Equation (II-185) the following relation

$$\begin{aligned} f_2^{(km)} &= f_0^{(km)} \int_0^{\varphi} \frac{f_1^{(km)}}{\bar{q}^2 f_0^{(km)}} d\varphi' = \frac{f_0^{(km)}}{kmi\omega^{(0)}} \int_0^{\varphi} f_1^{(km)} \frac{d\left(\frac{1}{f_0^{(km)}}\right)}{d\varphi'} d\varphi' \\ &= \frac{f_0^{(km)}}{kmi\omega^{(0)}} \left(\bar{c}^2 f_3^{(km)} - \frac{1}{2} (\bar{q}^2 - \bar{q}_{(0)}^2) \right) \end{aligned} \quad (II-187)$$

was used in the derivation of Equation (II-186). The relation given in Equation (II-186) holds for any value of φ . In particular, it holds at the nozzle entrance where it provides the admittance relation for the combustion chamber flow. The admittance relation given by Equation (II-186) is very general and can be specialized to obtain the solutions of the first, second or third order flows. In the latter two cases there are infinitely many such relations, one for each of the eigenfunctions that are present in the higher-order expansions. The summation of all of these admittance relations yields the complete second and third order Nozzle Admittance Relations. The above admittance relations apply to three dimensional flows.

By letting $S = 0$ identically the admittance relation, as given by Equation (II-186), can be used in the solution of homentropic flows. When the perturbed flow is irrotational, and homentropic, C_1 as well as σ must equal zero and consequently Equations (II-180) become: #

$$U - \frac{d}{d\varphi} \Phi_N = C_2 \frac{d}{d\varphi} \Phi_h$$

$$V - \Phi_N = C_2 \Phi_h$$

$$P + \bar{q}^2 U + k m i \omega'' \Phi_N - K = -C_2 k m i \omega'' \Phi_h \quad (\text{II-188})$$

The above equations are further simplified by observing that now the expressions for K and Φ_N are considerably simpler.

Eliminating C_2 from Equation (II-188) one obtains

$$\frac{U - \frac{d}{d\varphi} \Phi_N}{\frac{d}{d\varphi} \Phi_h} = \frac{V - \Phi_N}{\Phi_h} = \frac{P + \bar{q}^2 U + k m i \omega'' \Phi_N - K}{-k m i \omega'' \Phi_h}$$

$$U - \mu V = -\bar{c}^2 f_0^{(km)} \Gamma_N$$

$$(k m i \omega'' U + \mu P + \mu \bar{q}^2 U) = \mu K - k m i \omega'' \bar{c}^2 f_0^{(km)} \Gamma_N \quad (\text{II-189})$$

Since for irrotational flow $\frac{dV}{d\varphi} = U$, two of the following three equations are dependent. Using these three equations yields two admittance relations (i.e., Equation II-189) which can be shown to be identical.

which represent two admittance conditions when applied to the entrance of the nozzle. Observe that the last relation of Equation (II-189) can be obtained directly from Equation (II-186) by simply letting

$$S = \Gamma_v = \hat{Z} = 0.$$

In their present form Equations (II-186) through (II-189) are still quite general. To obtain the first order transverse admittance relation we must merely let Γ_N , M , K , \hat{Z} , Φ_N and $\frac{d}{d\phi}\Phi_N$ equal zero, wherever they appear in any one of these equations. Performing this operation results in the disappearance of the inhomogeneous parts of the above equations and the different cases of the first order, or linear, admittance relations are represented by a set of homogeneous relations.

To obtain the linear nozzle admittance relation for the case of purely axial oscillations the transverse components of the momentum equation as well as Equation (E-5) which is given in Appendix E must be eliminated from the previous analysis. In this case, the corresponding admittance relation can be obtained from Equations (II-180) by disregarding the second of those equations and all of the terms which are proportional to C_1 and by letting \hat{Z} , K , M , D , Φ_N and $\frac{d}{d\phi}\Phi_N$ equal zero. Making these simplifications and following the same procedure as the one that led to the derivation of Equation (II-186), one obtains the following admittance relation:

$$U(\bar{q}_f^2 \mu + k m i \omega'' U) + P \mu + S \bar{C}^2 (k m i \omega'' \Gamma_e - f_3^{(km)} \mu) = 0 \quad (\text{II-190})$$

For one-dimensional, isentropic oscillations the above admittance relation is replaced by

$$\frac{-k m i \omega'' U}{P + \bar{q}_f^2 U} = \mu$$

(II-191)

Note that this relation has the same form as one of the relations given in Equation (II-189). This similarity is, however, only formal since different equations are used to calculate the quantity μ (or Φ_h), which appears in these relations. Contrary to the case of transverse oscillations, in this case the second and third order admittance relations will not be derived. The analysis performed in this chapter was concerned with the behavior of finite amplitude continuous waves, and in the case of one-dimensional oscillations, it is most likely that the presence of nonlinear effects would result in the appearance of shock waves and consequently the nonlinear analysis performed in this section will not apply to this case.

Numerical Evaluations of the Solutions and of the Admittance Coefficients

In this section the differential equations controlling the behavior of μ and $\Gamma_x^{\#}$, which are defined in Equations (II-185a) and (185b) and appear in the definitions of the coefficients of the transverse admittance relations, will be derived and the method of obtaining their solution as well as the solution of Equation (II-175) for Φ will be discussed.

To obtain the solution of μ it is more convenient to transform the second order linear homogeneous ordinary differential equation for Φ_h (i.e., Equation II-176) into a Riccati equation for μ . Using the definition of μ and performing the suggested transformation results in the following first order nonlinear differential equation:

$$\frac{d}{d\varphi} \mu = \frac{1}{A} (B\mu - C) - \mu^2$$

(II-192)

where A , B and C are respectively the coefficients of the second, first and zeroth order derivatives of Φ , in Equation (II-175). Once μ is known, we can use its definition to obtain the solution of Φ_h .

For convenience in analysis all variables used in this section will be written without any superscripts and subscripts. The latter will be included only where required for clearer understanding of the analysis.

Since

$$\mu = \frac{1}{\Phi_h} \frac{d}{d\varphi} \Phi_h \quad (\text{II-193})$$

integration with respect to φ gives:

$$\Phi_h = \Phi_h(0) e^{\int_0^\varphi \mu(\varphi') d\varphi'} \quad (\text{II-194})$$

where $\Phi_h(0)$ has yet to be determined. Letting Φ_x (where x stands for either e , v or w) be the particular solution that satisfies Equation (II-175a) with either I_e , I_v or I_w present in its inhomogeneous part and using Equations (II-175a) and (II-176) gives:

$$\frac{1}{\bar{q}^2(\bar{c}^2 - \bar{q}^2)} \left(\Phi_x \mathcal{L}(\Phi_h) - \Phi_h \mathcal{L}(\Phi_x) \right) = \frac{d}{d\varphi} X$$

$$- \frac{1}{\bar{c}^2 - \bar{q}^2} \left(\frac{1}{\bar{c}^2} \frac{d\bar{q}^2}{d\varphi} + 2km i \omega \right) X = - \frac{\Phi_h I_x}{\bar{q}^2(\bar{c}^2 - \bar{q}^2)} \quad (\text{II-195})$$

where

$$X = \bar{c}^2 \int_0^{(km)} \Phi_h \Gamma_x \quad (\text{II-196})$$

Integrating Equation (II-195) gives:

$$X = e^{\beta(\varphi)} \left\{ X(\varphi_{\text{throat}}) - \int_{\varphi_{\text{throat}}}^\varphi \frac{\Phi_h I_x}{\bar{q}^2(\bar{c}^2 - \bar{q}^2)} e^{\beta(\varphi')} d\varphi' \right\} \quad (\text{II-197})$$

where, by using Equation (II-18) it can be shown that

$$\begin{aligned}\beta(\varphi) &= \int_{\varphi_{\text{throat}}}^{\varphi} \left(\frac{-\frac{d}{d\varphi} \left(\frac{\bar{q}^2}{\bar{c}^2} \right)}{1 - \frac{\bar{q}^2}{\bar{c}^2}} - \frac{2km\omega'''}{\bar{c}^2 - \bar{q}^2} \right) d\varphi' \\ &= \ln \left(1 - \frac{\bar{q}^2}{\bar{c}^2} \right) - \int_{\varphi_{\text{throat}}}^{\varphi} \frac{2km\omega'''}{\bar{c}^2 - \bar{q}^2} d\varphi'\end{aligned}\quad (\text{II-198})$$

Since

$$\lim_{\varphi \rightarrow \varphi_{\text{throat}}} \beta(\varphi) = -\infty \quad (\text{II-199})$$

then the only way to avoid a singularity of \mathbf{X} at the throat is to let $X(\varphi_{\text{throat}})$ equal zero. Using the definitions of \mathbf{X} and $f_o^{(km)}$ and Equations (II-194) and (II-197) it can be shown that:

$$\Gamma_x(\varphi) = -\frac{1}{\bar{c}^2 - \bar{q}^2} \int_{\varphi_{\text{throat}}}^{\varphi} \frac{I_x(\varphi')}{\bar{q}^2 \bar{c}^2 f_o^{(km)}(\varphi')} e^{\int_{\varphi}^{\varphi'} \left(\frac{2km\omega'''}{\bar{c}^2 - \bar{q}^2} + \frac{km\omega'''}{\bar{q}^2} - \mu \right) d\varphi''} d\varphi' \quad (\text{II-200})$$

Once the solution of μ is known, Γ_x can in principle be obtained from the above equation. However, in practice it is more convenient to solve for $\Gamma_x^\#$ by obtaining the solution for $(\bar{c}^2 - \bar{q}^2)\Gamma_x$ which, as can be checked by differentiation of Equation (II-200), is controlled by the following ordinary differential equation:

For further discussion of this point see Reference 5.

$$\frac{d}{d\varphi} \left((\bar{c}^2 - \bar{q}^2) \Gamma_x \right) + \left(\mu - \frac{2km\omega''}{\bar{c}^2 - \bar{q}^2} - \frac{km\omega''}{\bar{q}^2} \right) (\bar{c}^2 - \bar{q}^2) \Gamma_x$$

$$= - \frac{I_x}{f_0 \bar{c}^2 \bar{q}^2} \quad (\text{II-201})$$

If the calculation of the first order (or linear) admittance relation was the sole objective of this work, the integration of Equations (II-192) and (II-201) would provide us with all of the necessary information and there would be no need to solve for Φ . This is not the case when it is also desired to obtain the second and third order admittance relation whose calculation depends on the availability of the complete solution of the lower order equations. In other words, the knowledge of the solution of Φ'' is necessary for the calculation of the second order admittance relations and the knowledge of the solutions of Φ'' and Φ''' is necessary for the calculation of the third order admittance relations and so on. To obtain the complete solution of Φ it is first necessary to calculate Φ_x . Assuming that the solutions of Γ_x and μ are available, the solution of Φ_x can be obtained by a straightforward integration of Equation (II-185a). Performing this integration yields:

$$\Phi_x = e^{\int_0^\varphi \mu(\varphi') d\varphi'} \left\{ \Phi_{x(0)} - \int_0^\varphi \bar{c}^2 f_0^{(km)} \Gamma_x e^{-\int_0^{\varphi'} \mu(\varphi'') d\varphi''} d\varphi' \right\}$$

(II-202)

The first term on the right hand side of the above equation has exactly the same form as Φ_h and since the complete solution of Φ is the sum of the homogeneous and the particular solutions, this term

can be included in the homogeneous solution and eliminated from the definition of Φ_x , i.e.,

$$\Phi_x = - \int_0^\varphi \bar{c}^2 f_0^{(km)} \Gamma_x e^{-\int_0^{\phi'} \mu(\phi'') d\phi''} d\phi'$$

(II-203)

In practice it may be more convenient, however, to calculate Φ_x by solving instead the following first order ordinary differential

$$\frac{d}{d\varphi} \Phi_x - \mu \Phi_x = - \bar{c}^2 f_0^{(km)} \Gamma_x$$

(II-185a)

where $\Phi_x(0)$ is taken to be identically zero.

Once the solutions of Φ_x for $x = e, v$ and N are available and U, P, V and S are known at a given point then the constants C_1, σ and C_2 (or $\Phi_x(0)$), which appear in the definition of Φ , see Equation (II-179), can be calculated by use of any set of three linearly independent equations, out of the four given by Equation (II-180). The specialization of the above discussion to the cases in which the flow is homentropic or irrotational is trivial and will not be given here.

All the differential equations discussed in this section are complex and must be separated into their real and imaginary parts before proceeding with their solution. The nature of the coefficients of these equations precludes the possibility of obtaining their solutions analytically and numerical integration must be used instead. To obtain the solutions of these equations, which are regular at the throat[#], the integration must start at this point. To avoid the difficulties associated with the singularity at the throat, the differential equations are used to calculate several of the coefficients of the Taylor series expansions

[#] Note that the differential equations for μ as well as $(\bar{c}^2 - \bar{q}^2) \Gamma_x$ have a singularity at the throat.

of the regular solutions of these variables around the throat.[#] These series are used to provide the initial values of the dependent variables, which are necessary for the start of the numerical integration, at a point sufficiently removed from the singular point.

Combustion Chamber Flow

Before leaving this chapter it may be instructive^{##} to devote some space to the discussion of possible application of the theory derived in some of the previous sections, to the solution of the combustion chamber flow.

In particular, the equations and the solutions derived in this chapter could be used in analyzing the flow inside a combustion chamber for the special case in which all of the combustion is assumed to be concentrated inside an infinitesimally thin region. In this case, the flow conditions on both sides of the concentrated combustion zone (or on one side, if the concentrated combustion zone is assumed to be adjacent to the injector face) can be described by the same set of equations that was used in obtaining the solution of the nozzle flow. In analyzing the combustion chamber flow, it is convenient to let the reference quantities introduced in Equation (II-6) be the corresponding steady state quantities in either one of the two flow regimes. In the case in which the combustion zone is assumed to be concentrated at the injector face, this choice of reference quantities results in the following simplifications:

$$\bar{c} = \bar{p} = \bar{s} = \bar{p} = \bar{T} = 1$$

(II-204)

[#] Since Φ_x and $(\bar{c}^2 - \bar{q}^2)\Gamma_x$ are zero at the throat, the constant term of their expansion in Taylor series must be zero.

^{##} The comments presented in this section will be found quite useful for the understanding of some of the derivations presented in the next chapter; and this is the main reason for the introduction of this section here.

which holds exactly throughout the combustion chamber. It follows (see Reference 1) from this model of combustion that

$$\bar{q}_{c.c.} = \frac{\hat{q}}{c} = \text{Mach Number} = \text{Constant}$$

(II-205)

Substitution of Equations (II-204) and (II-205) into Equations (II-36) through (II-41) results in considerable simplification of these equations which reduce, after separation of variables,[#] to a set of ordinary differential equations with constant coefficients. The solutions of these equations, for the special case discussed above, will be obtained in Chapter IV where the analysis of the combustion chamber flow will be given in more detail.

[#] Eigenfunction expansion must still be employed in the solution of the second and third order equations.

CHAPTER III

DERIVATION OF THE NONLINEAR TRANSVERSE COMBUSTION-ZONE BOUNDARY CONDITION

Introduction

In the remainder of this thesis a detailed investigation of the behavior of nonlinear, transverse, periodic pressure waves inside the combustion chamber of liquid propellant rocket engines will be presented.

In order to simplify the analysis of the problem, the existence of a concentrated combustion zone at the injector end will be assumed. This results in the elimination of the source terms (mass, momentum and energy) from the equations describing the combustion chamber flow. This useful simplification, however, is partially counteracted by the need to replace the simple "solid wall" boundary condition (at the injector end) by another one representing the combustion process.

The derivation of this boundary condition will be presented in this chapter. The complete solution of the problem, which applies the results of this chapter as well as the results of the analysis of the nozzle flow to the determination of the nonlinear stability limits, and to the derivation of the expressions describing the behavior of finite amplitude pressure waves, will be presented in the following chapter.

Theoretical models, based on the existence of a concentrated combustion zone located at an arbitrary location of the combustion chamber, have been previously used in various studies of combustion problems. This model can be thought of as a limiting condition obtained in a rocket engine having a long combustion chamber and employing a particular injector design.[#]

Crocco¹, employing the time-lag concept, used this model in his studies of linear combustion instability. Sirignano⁴ extended Crocco's model to apply to the nonlinear case where the amplitudes of the pressure waves or applied disturbances are no longer small. Both of the above-mentioned works were limited to the study of one-dimensional wave oscillations. It is the purpose of this chapter to extend those

[#] Some injectors, such as the fuel-on-oxidizer injector or splash-plate injector, are characterized by high combustion efficiency which in turn results in low L^* requirements.

combustion models to apply to the study of the behavior of three-dimensional finite-amplitude waves.

Analysis

In the present study the concentrated combustion zone is assumed to be located in a plane immediately adjacent to the injector face. The mathematical boundary condition representing this limiting physical situation will be obtained by taking the limit, as the axial distance goes to zero, of the expressions which resulted from the application of the conservation laws to a differential volume element in the distributed combustion case. Following Crocco¹ the presence of liquid-propellant droplets in the combustion chamber will be replaced by a distribution of hot gas (combustion products) sources, whose behavior is time and space dependent.[#] Using this approximation, the application of the mass conservation law to a differential volume element in cylindrical coordinates,^{###} (see Figure 4) results in the following relation:

$$\begin{aligned} \frac{\Delta \rho}{\Delta t} \Delta z r \Delta \theta \Delta r + \frac{\Delta(\rho u r \Delta \theta \Delta r)}{\Delta z} \Delta z + \frac{\Delta(\rho v r \Delta \theta \Delta z)}{\Delta r} \Delta r \\ + \frac{\Delta(\rho w \Delta r \Delta z)}{r \Delta \theta} r \Delta \theta = Q \Delta z r \Delta \theta \Delta r \end{aligned} \quad \text{(III-1)}^{###}$$

where Δ is a difference operator, u , v and w are respectively the axial, radial and tangential components of velocity, ρ is the density of the gaseous combustion products and Q is the rate of gas generation per unit volume resulting from the burning of the liquid propellants. Dividing Equation (III-1) by $r \Delta \theta \Delta r$ and taking the

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- [#] This model of distributed combustion neglects momentum exchange due to droplet drag as well as energy exchange which results from the work done during the transformation from liquid drops into hot gases.
- ^{##} Although the equations describing the combustion chamber flow will be derived in the same coordinate system as used in the analysis of the nozzle flow, it is more convenient to use cylindrical coordinates in the present analysis. This will have no effect on the final results.
- ^{###} The variables appearing in this chapter are non-dimensionalized with respect to respective steady-state quantities. All the variables used in the analysis of this chapter are non-dimensional.

limit as $\Delta z \rightarrow 0$ (see Figure 5), which transforms the arbitrary volume element into a plane at some $z = \text{constant}$, yields the following expressions:

$$\Delta(\rho u) = \lim_{\Delta z \rightarrow 0} Q \Delta z \quad (\text{III-2})$$

where the right hand side of Equation (III-2) represents the mass generation per unit area. Equation (III-2) resulted from the particular limiting process performed and is no longer dependent on v and w , the transverse velocity components. It should, however, be emphasized that the variables appearing in (III-2) are still space (three-dimensional) and time-dependent. The additional conditions that must be satisfied by the flow-field variables on both sides of the concentrated combustion zone can be obtained by carrying similar limiting processes on the momentum (or vorticity) and energy (or entropy) equations. In the present study, however, the flow field is assumed to be irrotational, and the above suggested analysis of the momentum and energy equations can be replaced by the requirements that the gases generated at the concentrated combustion zone be irrotational and have constant entropy. Applying Equation (III-2) to a plane adjacent to the injector face ($z = 0$) gives the following relation

$$\Delta(\rho u) = \rho_2 u_2 - \rho_1 u_1 = \rho_2 u_2 = \rho u /_{z=0} = \lim_{\Delta z \rightarrow 0} Q \Delta z = \dot{m}_b \quad (\text{III-3})$$

since $u_1 = 0$ at the injector face; subscripts 1 and 2 represent the upstream and downstream sides of the concentrated combustion zone respectively. In Equation (III-3), $\rho_2 u_2$ represents mass flux of hot

gases leaving the concentrated combustion zone and $\dot{m}_b = \frac{dm_b}{dt}$ is the burning rate per unit area.

The exact form of \dot{m}_b will depend on the assumptions and expressions used in the description of the combustion process. In the present study Crocco's time-lag postulate,[#] which was successfully

[#] See Reference 1 for a complete discussion of the time-lag concept.

applied[#] to the study of linear combustion instability, will be employed. According to this postulate there is a time delay, $\tau(t)$, between the instant at which the liquid propellants are injected into the combustion chamber and the instant at which they are finally converted into hot combustion products. The length of this time delay is determined by the physical conditions in the combustion chamber which can, directly or indirectly, affect the burning rate. Oscillatory behavior of the physical factors (i.e., pressure, temperature, etc.) affecting the combustion process will result in a time-dependent time-lag that will oscillate about some average value. It then follows that propellant droplets burning at time t , were injected into the combustion chamber at time $t - \tau(t)$ where $\tau(t)$ (the time-lag) is the period necessary for the "conditioning" of the liquid propellants. If $t = 0$ designates the beginning of operation of a liquid propellant rocket motor, then the amount of propellants burned between $t = 0$ and $t = t$ was injected into the combustion chamber during the interval $t = 0$ to $t = t - \tau(t)$. Mathematically this can be expressed as follows:

$$\int_0^{t-\tau(t)} \frac{dm_i(t')}{dt'} dt' = \int_0^t \frac{dm_b(t')}{dt'} dt' \quad (\text{III-4})$$

Differentiating the above expression with respect to t yields:

$$\frac{dm_i(t-\tau(t))}{dt} \left(1 - \frac{d\tau(t)}{dt}\right) = \frac{dm_b(t)}{dt} \quad (\text{III-5})$$

where $\frac{dm_i}{dt}$ and $\frac{dm_b}{dt}$ are respectively the injection and burning rates of the liquid propellants. In studies of high frequency combustion instability it is frequently assumed that the injection-rate $\frac{dm_i}{dt}$, is a constant quantity independent of the conditions in the combustion chamber (this case is also known as Intrinsic Instability).

[#] See for example Reference 1 for the study of longitudinal instability and Reference 2 for studies of transverse instabilities.

Employing this assumption it follows that:

$$\frac{dm_i(t-\tau(t))}{dt} = \frac{dm_i(t)}{dt} = \frac{d\bar{m}_i}{dt} = \frac{d\bar{m}_b}{dt} \quad (\text{III-6})$$

Substituting Equation (III-6) into Equation (III-5) gives

$$\frac{dm_b(t)}{dt} = \frac{d\bar{m}_b}{dt} \left(1 - \frac{d\tau(t)}{dt}\right) \quad (\text{III-7})$$

which relates the burning rate $\frac{dm_b(t)}{dt}$ to the temporal variations of the time-lag. From Equation (III-7) it can be seen that the fluctuations

in the burning rate, $\frac{dm_b}{dt}$, are related to the fluctuations of the physical factors in the combustion chamber (upon which $\frac{d\tau(t)}{dt}$ is dependent).

Equation (III-7) is therefore a representation of the coupling between the combustion process and the oscillations of the physical factors in the combustion chamber. As will be seen later, Equation (III-7) is complex; hence the variables appearing in it must satisfy some definite phase and amplitude requirements. The time-lag can also be considered as a measure of the phase between the pressure and combustion oscillations. Crocco has shown that when the time-lag of the system satisfied certain requirements (i.e., $\tau(t) \sim (2n+1) \frac{T}{2}$ where $n = 0, 1, 2, 3 \dots$ and T is the period of the pressure oscillation), the phase is approximately zero. This is the optimum condition for linear instability.

The boundary condition derived in this section will be used in the investigation of the behavior of finite amplitude, periodic pressure waves. Since the waves are periodic, each of the expressions representing the waves generated at the concentrated combustion zone is proportional to $e^{im\omega t}$ where m is some integer and ω is an amplitude-dependent frequency. In this case, when t is the time, the period of oscillations, T , is an unknown. As in the analysis of the nozzle flow the transformation $y = \omega t$, where y represents a new time measure is introduced. Using the "new" time measure, the waves generated at the concentrated combustion zone are periodic with period equal to $\frac{2\pi}{m}$. Transforming Equation (III-7)

into the new time scale gives:

$$\frac{dm_b(y)}{dy} = \frac{d\bar{m}_b}{dy} \left(1 - \frac{dY}{dy} \right) \quad (\text{III-8})$$

where

$$Y = \omega \tau \quad (\text{III-9})$$

is the transformed time-lag. Combining Equations (III-9) and (III-3) gives:

$$\rho u|_{z=0} - \frac{d\bar{m}_b}{dy} \left(1 - \frac{dY}{dy} \right) = 0 \quad (\text{III-10})$$

which represents one of the boundary conditions that must be satisfied at the injector face.

Because of the nonlinearity of the equation describing the flow field inside the combustion chamber, a perturbation technique has been employed in their solution. According to this technique the unknowns appearing in the problem are assumed to have a power series expansion in some amplitude parameter ϵ . Since the unknowns of the problem must satisfy Equation (III-10) above, it will be convenient to develop it in a power series in ϵ . Only terms up to $O(\epsilon^3)$ will be necessary.[#]

Following Crocco, it is assumed that chemical reaction and final transformation of the liquid propellants into combustion products will take place after some entity E , which represents the necessary amount of droplet conditioning, has been accumulated. Mathematically E is defined as follows:

$$E = \int_{y-Y(y,\epsilon)}^y f(y',\epsilon) dy' = \int_{y-\bar{Y}(y,0)}^y \bar{f}(y',0) dy' \quad (\text{III-11})$$

where $f(y',\epsilon)$ represents the rate of accumulation of E , and the super-

[#] Solutions of the combustion chamber flow up to third order only, are necessary for the determination of the second-order eigenvalue perturbations which yield the nonlinear stability limit.

posed bar indicates a steady-state quantity. From Equation (III-11) it can be seen that the magnitude of $Y(y', \epsilon)$ depends on the nature of $f(y', \epsilon)$. For instance, if for $y - Y \leq y' \leq y$, $f(y', \epsilon)$ is larger than its average value, then $Y(y', \epsilon)$ will be lower than its average value since a shorter time will be necessary for the accumulation of E . The exact form of $f(y', \epsilon)$ is not known, but it is known to depend on the conditions (i.e., pressure, temperature, relative velocity between the propellants and hot gases, etc.) inside the combustion chamber. It will be convenient to continue the development with an arbitrary form of $f(y', \epsilon)$.

Employing the perturbation technique, E , f , Y , u and p are assumed to have the following Taylor series expansions:

$$E = \int_{y-Y(y,\epsilon)}^y f(y', \epsilon) dy' = \left[\int_{y-Y(y,\epsilon)}^y f(y', \epsilon) dy' \right]_{\epsilon=0} + \left[\frac{\partial}{\partial \epsilon} \int_{y-Y(y,\epsilon)}^y f(y', \epsilon) dy' \right]_{\epsilon=0} \epsilon + \frac{1}{2} \left[\frac{\partial^2}{\partial \epsilon^2} \int_{y-Y(y,\epsilon)}^y f(y', \epsilon) dy' \right]_{\epsilon=0} \epsilon^2 + \frac{1}{6} \left[\frac{\partial^3}{\partial \epsilon^3} \int_{y-Y(y,\epsilon)}^y f(y', \epsilon) dy' \right]_{\epsilon=0} \epsilon^3 + O(\epsilon^4) \quad (\text{III-12})$$

$$\frac{f(y', \epsilon)}{f_0} = 1 + f_1(y') \epsilon + f_2(y') \epsilon^2 + f_3(y') \epsilon^3 + O(\epsilon^4) \quad (\text{III-13a})$$

where f_0 is a steady-state, constant value of f and

$$f_i = \frac{1}{f_0} \left(\frac{\partial f(y', \epsilon)}{\partial \epsilon} \right)_{\epsilon=0}$$

$$f_2 = \frac{1}{2f_0} \left(\frac{\partial^2 f(y, \epsilon)}{\partial \epsilon^2} \right)_{\epsilon=0}$$

$$f_3 = \frac{1}{6f_0} \left(\frac{\partial^3 f(y, \epsilon)}{\partial \epsilon^3} \right)_{\epsilon=0}$$

(III-13b)

$$Y(y, \epsilon) = Y_0^{(0)} + Y_1(y)\epsilon + Y_2(y)\epsilon^2 + Y_3(y)\epsilon^3 + O(\epsilon^4)$$

(III-14a)

where

$$Y_0^{(0)} = (Y(y, \epsilon))_{\epsilon=0}$$

$$Y_1(y) = Y_0^{(1)} + Y_1^{(0)}(y) = Y_0^{(1)} + \tilde{Y}_1(y) = \left(\frac{\partial Y(y, \epsilon)}{\partial \epsilon} \right)_{\epsilon=0}$$

$$Y_2(y) = Y_0^{(2)} + Y_2^{(0)}(y) + Y_1^{(1)}(y) = Y_0^{(2)} + \tilde{Y}_2(y) = \frac{1}{2} \left(\frac{\partial^2 Y(y, \epsilon)}{\partial \epsilon^2} \right)_{\epsilon=0}$$

$$Y_3(y) = Y_0^{(3)} + Y_3^{(0)}(y) + Y_2^{(1)}(y) + Y_1^{(2)}(y) = Y_0^{(3)} + \tilde{Y}_3(y) = \frac{1}{6} \left(\frac{\partial^3 Y(y, \epsilon)}{\partial \epsilon^3} \right)_{\epsilon=0}$$

(III-14b)

$$u = \bar{q} + u^{(1)}\epsilon + u^{(2)}\epsilon^2 + u^{(3)}\epsilon^3 + O(\epsilon^4)$$

(III-15)

$$p = \bar{p} + p^{(1)}\epsilon + p^{(2)}\epsilon^2 + p^{(3)}\epsilon^3 + O(\epsilon^4) \quad (\text{III-16})$$

Examinations of the expressions defined in Equation (III-14b) shows that $Y_1(y)$, $Y_2(y)$ and $Y_3(y)$ can be written as a summation of a time-independent part and a time-dependent part.[#] The time-independent parts of these expressions are "eigenvalue"^{##} perturbations. They represent the correction to $Y_0^{(0)}$ which results from the presence of finite amplitude

[#] A clearer understanding of the notation used in Equation (III-14b) may possibly be obtained by rewriting Equation (III-14a) in the following equivalent form:

$$\begin{aligned} Y(y, \epsilon) = & Y_0^{(0)} + Y_1^{(1)}(y)\epsilon + Y_2^{(2)}(y)\epsilon^2 + Y_3^{(3)}(y)\epsilon^3 + O(\epsilon^4) \\ & + \epsilon [Y_0^{(1)} + Y_1^{(1)}(y)\epsilon + Y_2^{(1)}(y)\epsilon^2 + O(\epsilon^3)] \\ & + \epsilon^2 [Y_0^{(2)} + Y_1^{(2)}(y)\epsilon + O(\epsilon^2)] \\ & + \epsilon^3 [Y_0^{(3)} + O(\epsilon)] \end{aligned}$$

In this notation the superscript indicates the order of steady-state perturbation, and the subscript the order of the time dependent perturbation. The sum of the superscript and subscript will give the order of each term.

^{##} The word eigenvalue is written in quotations since $Y_0^{(j)}$ and $n^{(j)}$ (which will be introduced shortly) for $j = 0, 1, 2, \dots$, which appear in the boundary condition, are not eigenvalues in a true mathematical sense.

waves in the combustion chamber. The calculation of the values of these "eigenvalues" and the determination of their relationship with other eigenvalues of the problem (i.e., frequency and interaction index) are some of the objectives of this work.

Using Equations (III-11), (III-13a), (III-13b), (III-14a) and (III-14b) and application of Leibnitz's Rule in order to differentiate the integrals appearing in Equation (III-12) results in the following relation:

$$\begin{aligned}
 0 = & \left[\int_{y-Y_0^{(0)}}^y f_1(y') dy' + Y_1^{(1)}(y) \right] \epsilon + \left[\int_{y-Y_0^{(0)}}^y f_2(y') dy' \right. \\
 & \left. + f_1(y-Y_0^{(0)}) (Y_1^{(0)}(y) + Y_0^{(1)}) + Y_1^{(2)}(y) + Y_2^{(1)}(y) \right] \epsilon^2 \\
 & + \left[\int_{y-Y_0^{(0)}}^y f_3(y') dy' + f_2(y-Y_0^{(0)}) (Y_1^{(0)}(y) + Y_0^{(1)}) + f_1(y-Y_0^{(0)}) (Y_2^{(0)}(y) + Y_1^{(1)}(y) + Y_0^{(2)}) \right. \\
 & \left. - \frac{1}{2} \frac{df_1(y-Y_0^{(0)})}{dy} (Y_1^{(0)}(y) + Y_0^{(1)})^2 + Y_1^{(2)}(y) + Y_2^{(1)}(y) + Y_3^{(0)}(y) \right] \epsilon^3 + O(\epsilon^4) \quad (\text{III-17})
 \end{aligned}$$

In order to evaluate $\frac{dY}{dy}$ Equation (III-17) is rewritten in the following form:

$$\begin{aligned}
 - (\tilde{Y}_1(y) \epsilon + \tilde{Y}_2(y) \epsilon^2 + \tilde{Y}_3(y) \epsilon^3 + O(\epsilon^4)) = & \left[\int_{y-Y_0^{(0)}}^y f_1(y') dy' \right] \epsilon \\
 & + \left[\int_{y-Y_0^{(0)}}^y f_2(y') dy' + f_1(y-Y_0^{(0)}) (\tilde{Y}_1(y) + Y_0^{(1)}) \right] \epsilon^2 \\
 & + \left[\int_{y-Y_0^{(0)}}^y f_3(y') dy' + f_2(y-Y_0^{(0)}) (\tilde{Y}_1(y) + Y_0^{(1)}) + f_1(y-Y_0^{(0)}) (\tilde{Y}_2(y) + Y_0^{(2)}) \right. \\
 & \left. - \frac{1}{2} \frac{df_1(y-Y_0^{(0)})}{dy} (\tilde{Y}_1(y) + Y_0^{(1)})^2 \right] \epsilon^3 + O(\epsilon^4) \quad (\text{III-18})
 \end{aligned}$$

Successive substitution of the left side of Equation (III-18) into its right-hand side results in an explicit expression for $Y(y)$:

$$\begin{aligned}
 -Y(y) &= -(\tilde{Y}_1(y)\epsilon + \tilde{Y}_2(y)\epsilon^2 + \tilde{Y}_3(y)\epsilon^3 + O(\epsilon^4)) \\
 &= \left[\int_{y-Y_0^{(w)}}^y f_1(y') dy' \right] \epsilon + \left[\int_{y-Y_0^{(w)}}^y f_2(y') dy' + f_1(y-Y_0^{(w)}) \left(-\int_{y-Y_0^{(w)}}^y f_1(y') dy' + Y_0^{(w)} \right) \right] \epsilon^2 \\
 &\quad + \left[\int_{y-Y_0^{(w)}}^y f_3(y') dy' + f_2(y-Y_0^{(w)}) \left(-\int_{y-Y_0^{(w)}}^y f_1(y') dy' + Y_0^{(w)} \right) \right. \\
 &\quad \left. + f_1(y-Y_0^{(w)}) \left(-\int_{y-Y_0^{(w)}}^y f_2(y') dy' + f_1(y-Y_0^{(w)}) \left(\int_{y-Y_0^{(w)}}^y f_1(y') dy' - Y_0^{(w)} \right) + Y_0^{(2)} \right) \right. \\
 &\quad \left. - \frac{1}{2} \frac{df_1(y-Y_0^{(w)})}{dy} \left(-\int_{y-Y_0^{(w)}}^y f_1(y') dy' + Y_0^{(w)} \right)^2 \right] \epsilon^3 + O(\epsilon^4)
 \end{aligned} \tag{III-19}$$

Differentiating the above relation with respect to y (time) gives:

$$\begin{aligned}
 -\frac{dY(y)}{dy} &= -\left(\frac{d\tilde{Y}_1(y)}{dy} \epsilon + \frac{d\tilde{Y}_2(y)}{dy} \epsilon^2 + \frac{d\tilde{Y}_3(y)}{dy} \epsilon^3 \right) = (f_1(y) - f_1(y-Y_0^{(w)})) \epsilon \\
 &\quad + (f_2(y) - f_2(y-Y_0^{(w)}) - \frac{df_1(y-Y_0^{(w)})}{dy} \left(\int_{y-Y_0^{(w)}}^y f_1(y') dy' - Y_0^{(w)} \right) - f_1(y-Y_0^{(w)}) (f_1(y) - \\
 &\quad f_1(y-Y_0^{(w)})) \epsilon^2 + (f_3(y) - f_3(y-Y_0^{(w)}) - \frac{df_2(y-Y_0^{(w)})}{dy} \left(\int_{y-Y_0^{(w)}}^y f_1(y') dy' - Y_0^{(w)} \right) - f_2(y-Y_0^{(w)}) (f_1(y) \\
 &\quad - f_1(y-Y_0^{(w)})) - \frac{df_1(y-Y_0^{(w)})}{dy} \left(\int_{y-Y_0^{(w)}}^y f_2(y') dy' - 2f_1(y-Y_0^{(w)}) \left(\int_{y-Y_0^{(w)}}^y f_1(y') dy' - Y_0^{(w)} \right) - Y_0^{(2)} \right) \\
 &\quad - f_1(y-Y_0^{(w)}) (f_2(y) - f_2(y-Y_0^{(w)}) - f_1(y-Y_0^{(w)}) (f_1(y) - f_1(y-Y_0^{(w)})) \\
 &\quad - \frac{1}{2} \frac{d^2 f_1(y-Y_0^{(w)})}{dy^2} \left(-\int_{y-Y_0^{(w)}}^y f_1(y') dy' + Y_0^{(w)} \right)^2 + \frac{df_1(y-Y_0^{(w)})}{dy} \left(-\int_{y-Y_0^{(w)}}^y f_1(y') dy' + Y_0^{(w)} \right) (f_1(y) \\
 &\quad - f_1(y-Y_0^{(w)})) \epsilon^3 + O(\epsilon^4)
 \end{aligned} \tag{III-20}$$

Substituting the power series expansions (III-15), (III-16), and (III-20) of u , p and $\frac{dY(y)}{dy}$ into Equation (III-10) and separating the resulting equation according to powers of ϵ gives:

$$\bar{p} \bar{q} = \frac{d\bar{m}_b}{dy} \quad (\text{III-21})$$

$$\bar{p} u'' + p'' \bar{q} = - \frac{d\tilde{Y}_1(y)}{dy} \frac{d\bar{m}_b}{dy} \quad (\text{III-22})$$

$$\bar{p} u^{(2)} + p^{(2)} \bar{q} = - p'' u'' - \frac{d\tilde{Y}_2(y)}{dy} \frac{d\bar{m}_b}{dy} \quad (\text{III-23})$$

$$\bar{p} u^{(3)} + p^{(3)} \bar{q} = - p'' u^{(2)} - p^{(2)} u'' - \frac{d\tilde{Y}_3(y)}{dy} \frac{d\bar{m}_b}{dy} \quad (\text{III-24})$$

The above expressions are respectively the steady-state, first, second and third-order combustion zone boundary conditions and should be satisfied by the corresponding combustion chamber solutions. Note that $\frac{d\tilde{Y}_j}{dy}$ ($j = 1, 2, 3$) which appears in the above equation is still quite general as the exact form of $f(y, \epsilon)$ has not yet been specified.

Because of the complexity of the processes affecting it, the precise determination of $f(y, \epsilon)$ is presently an impossible task. Consequently, an alternative approach must be used. Following Crocco¹,

it will be assumed that the influence of the variations of the temperature upon the combustion process can be related to the variation of the pressure through some interaction index, n . (The effect of the relative velocity will not be considered in this work; for the study of this effect see Reference 2). Consequently $f(y', \epsilon)$ is assumed to have the following form

$$f(y, \epsilon) = P(y, \epsilon)^{n(\epsilon)} \quad \text{(III-25)} \quad \#$$

The main justification for using this form of $f(y, \epsilon)$ is its successful application in the treatment of "linear" problems (longitudinal as well as the transverse oscillations). The interaction index n , is the other "eigenvalue" of the problem and is assumed to have the following power series representation

$$n = n^{(0)} + n^{(1)}\epsilon + n^{(2)}\epsilon^2 + n^{(3)}\epsilon^3 + O(\epsilon^4) \quad \text{(III-26)}$$

Substitution of the power series representations of $P(y, \epsilon)$ (see Equation (II-14)) and $n(\epsilon)$ into Equation (III-25), logarithmic differentiation of the resulting expression and use of the definitions given in Equation (III-13b) yield:

$$f_1 = n^{(0)} \bar{p} \delta \tilde{\pi}^{(1)} \quad \text{(III-27)} \quad \#\#$$

$$f_2 = n^{(0)} \delta \bar{p} \tilde{\pi}^{(1)} + n^{(0)} (n^{(1)} - \frac{1}{2}) \delta^2 \bar{p}^{-2} \tilde{\pi}^{(1)2} + n^{(1)} \delta \bar{p} \tilde{\pi}^{(2)} \quad \text{(III-28)}$$

$$f_3 = (n^{(0)2} - \frac{3}{2} n^{(0)} n^{(1)}) \delta^2 \bar{p}^{-2} \tilde{\pi}^{(1)} \tilde{\pi}^{(1)} + n^{(1)} (\delta \bar{p} \tilde{\pi}^{(2)} + (n^{(0)} - \frac{1}{2}) \delta^2 \bar{p}^{-2} \tilde{\pi}^{(1)2}) \\ + (\frac{1}{3} n^{(0)} - \frac{1}{2} n^{(0)2} + \frac{1}{6} n^{(0)3}) \delta^3 \bar{p}^{-3} \tilde{\pi}^{(1)3} + n^{(2)} \delta \bar{p} \tilde{\pi}^{(1)} + n^{(0)} \delta \bar{p} \tilde{\pi}^{(3)} \quad \text{(III-29)}$$

where $\tilde{\pi}^{(j)}$ is defined in Equation (II-35). Substituting Equations (III-27) through (III-29) into Equations (III-20) through (III-24) gives the particular form, of the boundary condition, representing the concentrated combustion zone,

Similar analysis using a more general form of $f(y, \epsilon)$ is given in Reference 4.

In the following discussion a superposed "wobble" or the subscript c.c. will be used to denote quantities which describe the conditions inside the combustion chamber.

which must be satisfied by the solutions of this problem.

First Order Boundary Condition

Dividing Equation (III-22) by (III-21) and using (III-27) and (III-20) gives:

$$\begin{aligned}\tilde{\xi}''' + \tilde{\chi}''' &= - \frac{d\tilde{Y}_i(y)}{dy} \\ &= n'' \delta \bar{p} (\tilde{\pi}''(y) - \tilde{\pi}''(y - Y_0''))\end{aligned}\tag{III-30}$$

where $\tilde{\xi}^{(1)}$ and $\tilde{\chi}^{(1)}$ are defined in Equation (II-35). Equation (III-30) is the first order boundary condition, and it is analogous to the boundary condition used by Crocco in his "linear" analysis of the one dimensional problem. Although it is not indicated specifically, the variables appearing in (III-30) depend on all three space dimensions as well as on time.

The expressions for $\tilde{\xi}^{(1)}$, $\tilde{\chi}^{(1)}$, $\tilde{\pi}^{(1)}$, which appears in Equation (III-30) and $\eta_{c.c.}^{(1)}$, $\xi_{c.c.}^{(1)}$ which will be necessary for the derivations of the higher order boundary conditions are obtainable from the solution of the first order equations, describing the combustion chamber flow. Applying the simplification suggested in the last section of the previous chapter, these quantities can be obtained from Equations (II-84) through (II-90) and Equations (II-95a) through (II-95b). The solutions describing the irrotational flow field can be obtained from these equations by simply letting

$$\tilde{\omega}''' = \tilde{C}_i''' = 0$$

The above expression actually represents two additional boundary conditions which resulted from the assumption of the irrotationality of the flow and which replace the expressions that would have resulted from the application of a limiting process to the momentum and energy equations (see discussion in the beginning of this chapter).

Substitution of the expressions for $\tilde{\xi}^{(1)}$, $\tilde{\chi}^{(1)}$ and $\tilde{\pi}^{(1)}$

into Equation (III-30) and using the relation between $\tilde{\chi}^{(1)}$ and $\tilde{\pi}^{(1)}$ (for the case of irrotational flow) results in the derivation of the following boundary condition after separation of variables has been performed:

$$(1 - \bar{q}^2 \mathcal{W}^{(m)}) \frac{d}{dz} \tilde{\Phi}''' - im\omega^m \mathcal{W}^{(m)} \tilde{\Phi}''' = 0$$

(III-31) #, ##

This boundary condition will be used in Chapter IV where a detailed analysis of the combustion chamber flow will be presented.

Second Order Boundary Condition

The second and third order equations describing the wave propagation in the combustion chamber are linear, inhomogeneous partial differential equations. The solution of these equations is obtained by means of eigenfunction expansion. Using this method, the inhomogeneous parts of the second and third order equations are expanded in Fourier-type series in terms of the eigenfunctions satisfying the homogeneous parts of the respective equations. The application of this method results in the transformation of the original partial differential equation into an infinite number of independent ordinary differential equations, each of which is now amenable to a solution. The resulting solutions of the second and third order partial differential equations are expressed as an infinite series in terms of the appropriate eigenfunctions. Each of the terms appearing in these series must satisfy appropriate boundary conditions (i.e., the boundary conditions having the same transverse dependence which depends, of course, on the particular eigenfunction under consideration). Consequently similar expansion procedures will be performed in the analysis of the higher-order boundary conditions. As a result of the expansion procedures the higher-order boundary conditions will be expressed as an infinite series of eigenfunctions. Each term of this series will represent a separate boundary condition that will have to be satisfied by the corresponding eigenfunction in the series solution of the combustion chamber equations.

See Equation (III-36) for definition of $\mathcal{W}^{(m)}$.

To distinguish between the expressions describing the combustion chamber flow and these describing the nozzle flow the variable \mathbf{z} will replace ϕ (which is used in the nozzle analysis) in indicating the axial dependence of the solutions.

Although it was not stated explicitly, the variables appearing in Equations (III-1) through (III-30) are all real.[#] Since the solutions of the combustion chamber flow will be obtained in complex form, it will be more convenient to proceed with the derivation of the higher order boundary conditions by use of complex variables. Consequently, the second order combustion zone boundary condition will be rewritten as a complex expression. Substituting Equations (III-27) and (III-28) into (III-20) to get $\frac{d\tilde{Y}_2(y)}{dy}$ and using the latter in (III-23) gives, after dividing by $\bar{p}\bar{q}$:

$$\begin{aligned} \tilde{\xi}^{(2)} + \tilde{\chi}^{(2)} - n^{(0)}\gamma\bar{p}(\tilde{\pi}^{(2)}(y) - \tilde{\pi}^{(2)}(y - \gamma_0^{(0)})) &= -\frac{1}{2}\tilde{\chi}^{(0)}(\tilde{\xi}^{(1)} + \tilde{\xi}^{(1)*}) \\ + \frac{1}{2}n^{(0)2}\gamma^2\bar{p}\tilde{\pi}^{(1)}(y - \gamma_0^{(0)}) &(\tilde{\pi}^{(1)}(y - \gamma_0^{(0)}) + \tilde{\pi}^{(1)*}(y - \gamma_0^{(0)}) - \tilde{\pi}^{(1)}(y) - \tilde{\pi}^{(1)*}(y - \gamma_0^{(0)})) \\ + n^{(0)}(n^{(0)} - 1)\frac{1}{4}\bar{p}\gamma^2 &(\tilde{\pi}^{(1)}(y)(\tilde{\pi}^{(1)}(y) + \tilde{\pi}^{(1)*}(y)) \\ - \tilde{\pi}^{(1)}(y - \gamma_0^{(0)}) &(\tilde{\pi}^{(1)}(y - \gamma_0^{(0)}) + \tilde{\pi}^{(1)*}(y - \gamma_0^{(0)})) \\ - \bar{p}^2\gamma^2n^{(0)2} \frac{\partial\tilde{\pi}^{(1)}(y - \gamma_0^{(0)})}{\partial y} \int_{y - \gamma_0^{(0)}}^y &(\tilde{\pi}^{(1)}(y') + \tilde{\pi}^{(1)*}(y'))dy' + n^{(0)}\gamma\bar{p} \frac{\partial\tilde{\pi}^{(1)}(y - \gamma_0^{(0)})}{\partial y} \gamma_0^{(0)} \\ + \gamma\bar{p}n^{(0)} &(\tilde{\pi}^{(1)}(y) - \tilde{\pi}^{(1)}(y - \gamma_0^{(0)})) \end{aligned} \quad (\text{III-32})$$

where now $\tilde{\pi}^{(j)}$, $\tilde{\chi}^{(j)}$ and $\tilde{\xi}^{(j)}$ for $j = 1, 2$ are complex quantities and an asterick superscript indicates the complex conjugate of the quantity.

[#] With the exception of Equation (III-30) which has the same form in the complex or real notation.

The following relations are available from the second order analysis of the combustion chamber flow:

$$\tilde{\xi}^{(2)} = \frac{\partial}{\partial z} \tilde{F}^{(2)}$$

$$\tilde{\chi}^{(2)} = \tilde{G}^{(2)} + \tilde{\pi}^{(2)}$$

$$\tilde{\pi}^{(2)} = \tilde{A}^{(2)} - \omega'' \frac{\partial}{\partial y} \tilde{F}^{(2)} - \omega'' \frac{\partial}{\partial y} \tilde{F}^{(2)} - \bar{q}^2 \frac{\partial}{\partial z} \tilde{F}^{(2)}$$

(III-33)

where

$$\tilde{A}^{(2)} = \frac{1}{2} \tilde{\pi}^{(1)2} - \frac{1}{2} \bar{q}^2 \tilde{\xi}^{(1)2} - \psi \bar{p} \bar{q} \eta_{c.c.}^{(1)2} - \frac{1}{4} \psi \bar{p} \bar{q} \xi_{c.c.}^{(1)2}$$

and

$$\tilde{G}^{(2)} = -\frac{1}{2} (r-1) \tilde{\pi}^{(1)2}$$

All the quantities appearing in the above equations, which are applicable to the irrotational case, are real. Rewriting these relations in complex notation and substituting them together with the expressions defined in Equation (III-31) into Equation (III-32) gives:

$$(1 - \bar{q}^2) \frac{\partial}{\partial z} \tilde{F}^{(2)} + \bar{q}^2 \eta'' r \left(\frac{\partial}{\partial z} \tilde{F}^{(2)} - \frac{\partial}{\partial z} \tilde{F}^{(2)} - \gamma'' \right)$$

$$- \omega'' \left(\frac{\partial}{\partial y} \tilde{F}^{(2)} - \eta'' r \left(\frac{\partial}{\partial y} \tilde{F}^{(2)} - \frac{\partial}{\partial y} \tilde{F}^{(2)} - \gamma'' \right) \right)$$

(III-32a)

$$= \hat{I}^{(2)}$$

where #

$$\begin{aligned} \hat{I}^{(2)} = & \left[\frac{1}{4} (\gamma - 1 - \omega^{(2m)}) \tilde{p}^{(1)2} + \frac{1}{4} \bar{g}^2 \omega^{(2m)} \tilde{u}^{(1)2} - \frac{1}{2} \tilde{R}^{(1)} \tilde{u}^{(1)} \right. \\ & - n^{(1)2} \gamma^2 e^{(m)} g^{(m)} \tilde{p}^{(1)2} + \frac{1}{4} n^{(1)} (n^{(1)} - 1) \gamma^2 g^{(2m)} \tilde{p}^{(1)2} \left. \right] \Theta^{(1)2} e^{2im\eta} J_\nu^2(s_{(1,h)} \sqrt{\frac{\psi'}{\psi_w}}) \\ & + \left[\frac{1}{4} (\gamma - 2) \tilde{p}^{(1)} \tilde{p}^{(1)*} + \frac{1}{4} \bar{g}^2 \tilde{u}^{(1)} \tilde{u}^{(1)*} - \frac{1}{2} \tilde{R}^{(1)} \tilde{u}^{(1)*} \right] \Theta^{(1)} \Theta^{(1)*} J_\nu^2(s_{(1,h)} \sqrt{\frac{\psi'}{\psi_w}}) \\ & + \left[\tilde{V}^{(1)2} \Theta^{(1)2} e^{2im\eta} \omega^{(2m)} + \tilde{V}^{(1)} \tilde{V}^{(1)*} \Theta^{(1)} \Theta^{(1)*} \right] \bar{g} \frac{1}{2} \psi \left(\frac{d}{d\psi} J_\nu(s_{(1,h)} \sqrt{\frac{\psi'}{\psi_w}}) \right)^2 \\ & + \left[\tilde{V}^{(1)2} \left(\frac{d}{d\theta} \Theta^{(1)} \right)^2 e^{2im\eta} \omega^{(2m)} \right. \\ & + \left. \tilde{V}^{(1)} \tilde{V}^{(1)*} \left(\frac{d}{d\theta} \Theta^{(1)} \right) \left(\frac{d}{d\theta} \Theta^{(1)*} \right) \right] \bar{g} \frac{1}{8\psi} J_\nu^2(s_{(1,h)} \sqrt{\frac{\psi'}{\psi_w}}) \\ & + (in^{(1)} \gamma_0 e^{(m)} + g^{(m)} n^{(1)}) \gamma \tilde{p}^{(1)} \Theta^{(1)} J_\nu(s_{(1,h)} \sqrt{\frac{\psi'}{\psi_w}}) \bar{J}_\nu(s_{(1,h)}) e^{im\eta} \\ & + i\omega^{(1)} \omega^{(m)} \tilde{\Phi}^{(1)} \Theta^{(1)} J_\nu(s_{(1,h)} \sqrt{\frac{\psi'}{\psi_w}}) e^{im\eta} \end{aligned}$$

It is now desired to expand the right side of Equation (III-32a) in terms of the appropriate eigenfunctions. Using the appropriate form of the first order solution (i.e. standing or travelling wave solution), the boundary condition to be satisfied by both types of the second order solutions will now be developed. Substituting the appropriate trigonometric identities, and the expansions developed in Appendix C, into Equation (III-32a) and following the same procedures as in the analysis of the nozzle flow gives:

See Equation (III-36) for the definitions of $\omega^{(m)}$, $g^{(m)}$ and $e^{(m)}$ which appear in the definition of $\hat{I}^{(2)}$

$$\begin{aligned}
 & \sum_{m_2, n_2, q} \left\{ (1 - \bar{q}^2 \mathbb{W}^{(m_2)}) \frac{d}{dz} \tilde{F}_{(m_2, n_2, q)}^{(2)} - i m_2 \omega^{(n)} \mathbb{W}^{(m_2)} \tilde{F}_{(m_2, n_2, q)}^{(2)} \right\} \\
 &= \sum_{q=1}^{\infty} \left\{ Q_{(2m, 2\nu, q)}^{(2)} e^{2im\gamma} + Q_{(0, 2\nu, q)}^{(2)} \right\} \cos 2\nu\theta J_{2\nu}(S_{(2\nu, q)} \sqrt{\frac{\Psi'}{\Psi_w}}) \\
 &+ \sum_{q=0}^{\infty} \left\{ Q_{(2m, 0, q)}^{(2)} e^{2im\gamma} + Q_{(0, 0, q)}^{(2)} \right\} J_0(S_{(0, q)} \sqrt{\frac{\Psi'}{\Psi_w}}) \quad (\text{III-34})^\# \\
 &+ \left\{ (iY_0^{(n)} n^{(n)} e^{(m)} + n^{(n)} g^{(m)}) \gamma \tilde{p}^{(n)} + i\omega^{(n)} \mathbb{W}^{(m)} \tilde{\Phi}^{(n)} \right\} \cos \nu\theta J_\nu(S_{(\nu, n)} \sqrt{\frac{\Psi'}{\Psi_w}}) e^{im\gamma}
 \end{aligned}$$

for the case of standing waves and

$$\begin{aligned}
 & \sum_{m_2, n_2, q} \left\{ (1 - \bar{q}^2 \mathbb{W}^{(m_2)}) \frac{d}{dz} \tilde{F}_{(m_2, n_2, q)}^{(2)} - i m_2 \omega^{(n)} \mathbb{W}^{(m_2)} \tilde{F}_{(m_2, n_2, q)}^{(2)} \right\} \\
 &= \sum_{q=1}^{\infty} 2Q_{(2m, 2\nu, q)}^{(2)} e^{2i(m\gamma + \nu\theta)} J_{2\nu}(S_{(2\nu, q)} \sqrt{\frac{\Psi'}{\Psi_w}}) + \sum_{q=0}^{\infty} 2Q_{(0, 0, q)}^{(2)} J_0(S_{(0, q)} \sqrt{\frac{\Psi'}{\Psi_w}}) \quad (\text{III-35}) \\
 &+ \left\{ (iY_0^{(n)} n^{(n)} e^{(m)} + n^{(n)} g^{(m)}) \gamma \tilde{p}^{(n)} + i\omega^{(n)} \mathbb{W}^{(m)} \tilde{\Phi}^{(n)} \right\} e^{i(m\gamma + \nu\theta)} J_\nu(S_{(\nu, n)} \sqrt{\frac{\Psi'}{\Psi_w}})
 \end{aligned}$$

for the case of travelling waves. $Q_{(m_2, n_2, q)}^{(2)}$, $\mathbb{W}_j^{(m)}$ and $\tilde{F}_{(m_2, n_2, q)}^{(2)}$

which appear in Equation (III-34) and (III-35) have the following definitions:

[#] The exact value of the indices m_2 , n_2 and q is determined by the expansion form of the inhomogeneous side of this expression.

$$\psi^{(m_j)} = 1 - \gamma n^{(0)} g^{(m_j)}$$

$$g^{(m_j)} = 1 - e^{(m_j)}$$

(III-36) #

$$e^{(m_j)} = e^{-im_j Y_o^{(0)}}$$

$$\tilde{F}_{(m_1, n_1, \nu, \gamma)_S}^{(2)} = \tilde{\Phi}_{(m_1, n_1, \nu, \gamma)_S}^{(2)} \cos n_1 \nu \theta J_{n_1 \nu}(\zeta_{n_1 \nu} \sqrt{\frac{\psi}{\psi_0}}) e^{im_1 \gamma} \text{ (for standing waves)}$$

(III-37a)

and

$$\tilde{F}_{(m_2, n_2, \nu, \gamma)_T}^{(2)} = \tilde{\Phi}_{(m_2, n_2, \nu, \gamma)_T}^{(2)} e^{i(n_2 \nu \theta + m_2 \gamma)} J_{n_2 \nu}(\zeta_{n_2 \nu} \sqrt{\frac{\psi}{\psi_0}})$$

(III-37b)

(for travelling waves)

Both of the above expressions represent typical eigenfunctions which satisfy the second order equations.

$$Q_{(2m, 2\nu, \gamma)}^{(2)} = \left(\frac{1}{4} (\gamma - 1 - \psi^{(2m)}) \tilde{P}^{(1)2} + \frac{1}{4} \bar{q}^2 \psi^{(2m)} \tilde{u}^{(1)2} - \frac{1}{2} \tilde{R}^{(1)} \tilde{u}^{(1)} \right.$$

(III-38)

$$\left. - n^{(1)2} \gamma^2 e^{(m)} g^{(m)} \tilde{P}^{(1)2} + \tilde{P}^{(1)2} \frac{1}{4} n^{(1)} (n^{(1)} - 1) \gamma^2 g^{(2m)} \right) \frac{1}{2} A_{(2\nu, \gamma)} + \frac{1}{4\psi_0} \tilde{V}^{(1)2} \bar{q} \psi^{(2m)} (B_{(2\nu, \gamma)} - \nu^2 C_{(2\nu, \gamma)})$$

$$Q_{(2m, 0, \gamma)}^{(2)} = \left(\frac{1}{4} (\gamma - 1 - \psi^{(2m)}) \tilde{P}^{(1)2} + \frac{1}{4} \bar{q}^2 \psi^{(2m)} \tilde{u}^{(1)2} - \frac{1}{2} \tilde{R}^{(1)} \tilde{u}^{(1)} - n^{(1)2} \gamma^2 e^{(m)} g^{(m)} \tilde{P}^{(1)2} \right.$$

$$\left. + \frac{1}{4} n^{(1)} (n^{(1)} - 1) \gamma^2 g^{(2m)} \tilde{P}^{(1)2} \right) \frac{1}{2} A_{(0, \gamma)} + \frac{1}{4\psi_0} \tilde{V}^{(1)2} \bar{q} \psi^{(2m)} (B_{(0, \gamma)} + \nu^2 C_{(0, \gamma)})$$

(III-39)

ψ is a Hebrew letter pronounced "shin".

$$Q_{(0,0,q)}^{(2)} = \left(\frac{1}{4}(r-2)\tilde{P}^{(0)}\tilde{P}^{(0)*} + \frac{1}{4}\bar{q}^2\tilde{u}^{(0)}\tilde{u}^{(0)*} - \frac{1}{2}\tilde{R}^{(0)}\tilde{u}^{(0)*} \right) \frac{1}{2}A_{(0,q)} + \frac{1}{4\psi_w}\tilde{V}^{(0)}\tilde{V}^{(0)*}\bar{q} (B_{(0,q)} - \nu^2 C_{(0,q)}) \quad (\text{III-40})$$

$$Q_{(0,0,q)}^{(2)} = \left(\frac{1}{4}(r-2)\tilde{P}^{(0)}\tilde{P}^{(0)*} + \frac{1}{4}\bar{q}^2\tilde{u}^{(0)}\tilde{u}^{(0)*} - \frac{1}{2}\tilde{R}^{(0)}\tilde{u}^{(0)*} \right) \frac{1}{2}A_{(0,q)} + \frac{1}{4\psi_w}\tilde{V}^{(0)}\tilde{V}^{(0)*}\bar{q} (B_{(0,q)} + \nu^2 C_{(0,q)}) \quad (\text{III-41})$$

and

$$\psi_w = \frac{1}{2}\bar{P}\bar{q}r_w^2 = \frac{1}{2}\bar{q} \quad (\text{III-42})$$

Since in the combustion chamber $r_w = \bar{P} = 1$. Finally, substituting Equation (III-37a) into (III-34), (III-37b) into (III-35) and separating variables results in the following forms of the boundary condition that must be satisfied by each of the terms appearing in the series solution of the second order equations:

$$(1 - \bar{q}^2 \mathcal{W}^{(m_2)}) \frac{d}{dz} \tilde{\Phi}_{(m_2, n_2, q)}^{(2)} - i m_2 \omega^{(n_2)} \mathcal{W}^{(m_2)} \tilde{\Phi}_{(m_2, n_2, q)}^{(2)} = Q_{(m_2, n_2, q)}^{(2)} \quad (\text{III-43})$$

for the standing wave solution and

$$(1 - \bar{q}^2 \mathcal{W}^{(m_2)}) \frac{d}{dz} \tilde{\Phi}_{(m_2, n_2, q)}^{(2)} - i m_2 \omega^{(n_2)} \mathcal{W}^{(m_2)} \tilde{\Phi}_{(m_2, n_2, q)}^{(2)} = 2 Q_{(m_2, n_2, q)}^{(2)} \quad (\text{III-44})$$

for the travelling wave solution.

The subscripts (m_2, n_2, q) take on the same values as the corresponding subscripts appearing in the second order solution of the combustion chamber equations.

Third Order Boundary Condition

The derivation of the third-order boundary condition will follow a procedure similar to the one used in the derivation of the second-order boundary condition. Evaluating $-\frac{dY_3(y)}{dy}$ by use of Equations (III-27), (III-28), (III-29) and (III-20) and using it in (III-24) gives:

$$\begin{aligned}
 \tilde{\xi}^{(2)} + \tilde{\chi}^{(1)} \tilde{\xi}^{(2)} + \tilde{\chi}^{(2)} \tilde{\xi}^{(1)} + \tilde{\chi}^{(3)} &= A_1 (\tilde{\pi}^{(2)}(y) \tilde{\pi}^{(1)}(y) - \tilde{\pi}^{(2)}(y - Y_0^{(1)}) \tilde{\pi}^{(1)}(y - Y_0^{(1)})) \\
 &+ A_2 (\tilde{\pi}^{(1)2}(y) - \tilde{\pi}^{(1)2}(y - Y_0^{(1)})) + n^{(1)} \gamma (\tilde{\pi}^{(1)}(y) - \tilde{\pi}^{(1)}(y - Y_0^{(1)})) + n^{(2)} \gamma (\tilde{\pi}^{(1)}(y) \\
 &- \tilde{\pi}^{(1)}(y - Y_0^{(1)})) - [n^{(1)} \gamma (A_3 \tilde{\pi}^{(1)2}(y - Y_0^{(1)}) + n^{(1)} \gamma \tilde{\pi}^{(2)}(y - Y_0^{(1)})) \\
 &+ n^{(1)3} \gamma^3 \frac{\partial \tilde{\pi}^{(1)}(y - Y_0^{(1)})}{\partial y} \int_{y - Y_0^{(1)}}^y \tilde{\pi}^{(1)}(y') \partial y' - n^{(1)3} \gamma^3 \tilde{\pi}^{(1)2}(y - Y_0^{(1)})] (\tilde{\pi}^{(1)}(y) - \tilde{\pi}^{(1)}(y - Y_0^{(1)})) \\
 &- \left[\left(A_3 \frac{\partial \tilde{\pi}^{(1)2}(y - Y_0^{(1)})}{\partial y} + n^{(1)} \gamma \frac{\partial \tilde{\pi}^{(2)}(y - Y_0^{(1)})}{\partial y} \right) n^{(1)} \gamma - n^{(1)3} \gamma^3 \frac{\partial \tilde{\pi}^{(1)2}(y - Y_0^{(1)})}{\partial y} \right] \int_{y - Y_0^{(1)}}^y \tilde{\pi}^{(1)}(y') \partial y' \\
 &- n^{(1)} \gamma \frac{\partial \tilde{\pi}^{(1)}(y - Y_0^{(1)})}{\partial y} \int_{y - Y_0^{(1)}}^y (A_3 \tilde{\pi}^{(1)2}(y') + n^{(1)} \gamma \tilde{\pi}^{(2)}(y')) \partial y' - n^{(1)} \gamma \tilde{\pi}^{(1)}(y - Y_0^{(1)}) (A_3 \tilde{\pi}^{(1)2}(y) \\
 &+ n^{(1)} \gamma \tilde{\pi}^{(2)}(y) - A_3 \tilde{\pi}^{(1)2}(y - Y_0^{(1)}) - n^{(1)} \gamma \tilde{\pi}^{(2)}(y - Y_0^{(1)})) \\
 &- \frac{1}{2} n^{(1)3} \gamma^3 \frac{\partial^2 \tilde{\pi}^{(1)}(y - Y_0^{(1)})}{\partial y^2} \left(\int_{y - Y_0^{(1)}}^y \tilde{\pi}^{(1)}(y') \partial y' \right)^2 + n^{(1)} \gamma \frac{\partial \tilde{\pi}^{(1)}(y - Y_0^{(1)})}{\partial y} Y_0^{(2)} \\
 &= \hat{I}^{(3)}
 \end{aligned}$$

(III-45)

where

$$A_1 = n^{(0)2} \gamma^2 - \frac{2}{3} n^{(0)} \gamma^2$$

$$A_2 = \frac{1}{3} n^{(0)} - \frac{1}{2} n^{(0)2} + \frac{1}{6} n^{(0)3} \quad (\text{III-46})$$

$$A_3 = \frac{1}{2} n^{(0)} (n^{(0)} - 1) \gamma^2$$

By use of the second-order analysis of the combustion chamber flow, it will be shown that $\omega^{(1)} = n^{(1)} = Y_o^{(1)} = 0$. Consequently the terms proportional to these quantities are missing from Equation (III-45). Using the relations between $\tilde{\xi}^{(3)}$, $\tilde{\pi}^{(3)}$, $\tilde{\chi}^{(3)}$ and $\tilde{F}^{(3)}$, # Equation (III-45) can be rewritten in the following form:

The following relations are available from the analysis of the combustion chamber flow (for the case of irrotational flow):

$$\tilde{\xi}^{(3)} = \frac{\partial}{\partial z} \tilde{F}^{(3)}$$

$$\tilde{\chi}^{(3)} = \tilde{G}^{(3)} + \tilde{\pi}^{(3)}$$

$$\tilde{\pi}^{(3)} = \tilde{A}^{(3)} - \omega^{(2)} \frac{\partial}{\partial y} \tilde{F}^{(1)} - \omega^{(0)} \frac{\partial}{\partial y} \tilde{F}^{(3)} - \bar{q}^2 \frac{\partial}{\partial z} \tilde{F}^{(3)}$$

where

$$\tilde{A}^{(3)} = \left(\frac{1}{2} (r-1) - \frac{1}{6} (r^2-1) \right) \bar{q}^2 \tilde{\chi}^{(0)3} + \frac{1}{3} (r-2) \tilde{\chi}^{(0)3} + \tilde{\chi}^{(0)} \tilde{\chi}^{(0)} - \bar{q}^2 \tilde{\xi}^{(0)} \tilde{\xi}^{(2)}$$

$$- 2 \bar{F} \bar{q} \psi \eta_{c.c}^{(0)} \eta_{c.c}^{(0)} - \frac{1}{2} \frac{\bar{F} \bar{q}}{\psi} \mathcal{S}_{c.c}^{(0)} \mathcal{S}_{c.c}^{(2)}$$

and

$$\tilde{G}^{(3)} = \frac{1}{3} \gamma^2 \tilde{\pi}^{(0)2} - \gamma \tilde{\pi}^{(0)} \tilde{\pi}^{(2)} - \frac{1}{3} \tilde{\chi}^{(0)3} + \tilde{\chi}^{(0)} \tilde{\chi}^{(2)}$$

All the variables appearing in the above expressions are real. For the expansion form of the second order solutions see Appendix B or the second order analysis of the nozzle flow which was presented in the previous chapter.

$$\begin{aligned}
 & (1 - \bar{q}^2) \frac{\partial}{\partial z} \tilde{F}^{(3)} - \omega^{(0)} \frac{\partial}{\partial y} \tilde{F}^{(3)} + n^{(0)} \gamma \bar{q}^2 \left(\frac{\partial}{\partial z} \tilde{F}^{(3)}(y) - \frac{\partial}{\partial z} \tilde{F}^{(3)}(y - Y_0^{(0)}) \right) \\
 & + n^{(0)} \omega^{(0)} \gamma \left(\frac{\partial}{\partial y} \tilde{F}^{(3)}(y) - \frac{\partial}{\partial y} \tilde{F}^{(3)}(y - Y_0^{(0)}) \right) = - \tilde{G}^{(3)} + \omega^{(2)} \frac{\partial}{\partial y} \tilde{F}^{(3)} \\
 & - \tilde{A}^{(3)} + n^{(0)} \gamma \left(A^{(3)}(y) - A^{(3)}(y - Y_0^{(0)}) \right) - \omega^{(2)} \left(\frac{\partial}{\partial y} \tilde{F}^{(3)}(y) \right. \\
 & \left. - \frac{\partial}{\partial y} \tilde{F}^{(3)}(y - Y_0^{(0)}) \right) - \tilde{X}^{(3)} \tilde{\xi}^{(2)} - \tilde{X}^{(2)} \tilde{\xi}^{(3)} + \hat{I}^{(3)}
 \end{aligned}$$

(III-47)

Equation (III-47) must now be rewritten in complex form and then expanded in terms of the appropriate eigenfunctions. With the exception of $\hat{I}^{(3)}$, which is the inhomogeneous part of Equation (III-45), the transformation into complex variables and the expansion of the terms appearing in the inhomogeneous part of Equation (B-47) is quite straightforward (see Appendices A and B for illustrations of the methods used in the manipulation of these terms) and will not be given here. In the following section the terms appearing in $\hat{I}^{(3)}$ will be rewritten in complex notation. In what follows, let

$$\begin{aligned}
 [a] &= a(y) - a(y - Y_0^{(0)}) \quad \text{where } a \text{ is an arbitrary quantity} \\
 A_4 &= n^{(0)} \gamma A_3 - n^{(0)3} \gamma^3 \\
 A_5 &= \frac{1}{2}(\gamma - 1)
 \end{aligned}$$

(III-48)

Then we get, using the relations between $\tilde{\pi}^{(2)}$ and $\tilde{\chi}^{(2)}$, $\tilde{\pi}^{(1)}$ and $\tilde{\chi}^{(1)}$:

$$A_1[\tilde{\pi}^{(2)}\tilde{\pi}^{(1)}] = A_1\left[\frac{1}{2}\tilde{\chi}^{(2)}(\tilde{\chi}^{(1)} + \tilde{\chi}^{(1)*})\right] + A_1A_5\frac{1}{4}(g^{(3m)}\tilde{\chi}^{(1)3} + 3\tilde{\chi}^{(1)2}\tilde{\chi}^{(1)*}g^{(m)})$$

(III-49) #

$$A_2[\tilde{\chi}^{(1)3}] = \frac{1}{4}A_2(\tilde{\chi}^{(1)3}g^{(3m)} + 3\tilde{\chi}^{(1)2}\tilde{\chi}^{(1)*}g^{(m)})$$

$$-A_4\tilde{\pi}^{(1)2}(y-Y_0^{(w)})[\tilde{\pi}^{(1)}] = -\frac{1}{4}A_4(\tilde{\chi}^{(1)3}e^{(2m)}g^{(m)} + (2g^{(m)} + e^{(2m)}g^{(m)*})\tilde{\chi}^{(1)2}\tilde{\chi}^{(1)*})$$

(III-50)

(III-51)

$$-n^{(w)2}\gamma^2\tilde{\pi}^{(2)}(y-Y_0^{(w)})[\tilde{\pi}^{(1)}] = -n^{(w)2}\gamma^2\left\{\frac{1}{2}\tilde{\chi}^{(2)}(y-Y_0^{(w)})(\tilde{\chi}^{(1)}g^{(m)} + \tilde{\chi}^{(1)*}g^{(m)*})\right.$$

$$\left. + \frac{1}{4}A_5(\tilde{\chi}^{(1)3}e^{(2m)}g^{(m)} + (2g^{(m)} + e^{(2m)}g^{(m)*})\tilde{\chi}^{(1)2}\tilde{\chi}^{(1)*}\right\}$$

(III-52)

$$-n^{(w)2}\gamma^3\left(\frac{\partial}{\partial y}\tilde{\pi}^{(1)}(y-Y_0^{(w)})\right)\left(\int_{y-Y_0^{(w)}}^y\tilde{\pi}^{(1)}(y')\partial y'\right)[\tilde{\pi}^{(1)}] = -\frac{1}{4}n^{(w)3}\gamma^3\left\{\tilde{\chi}^{(1)3}e^{(m)}g^{(m)2}\right.$$

$$\left.- e^{(m)*}g^{(m)2}\tilde{\chi}^{(1)2}\tilde{\chi}^{(1)*}\right\}$$

(III-54)

The product appearing inside the square brackets should be evaluated following the procedure outlined in Appendix B. The resulting expression should then be evaluated at y and $y - Y_0^{(o)}$ and the operation indicated by the square brackets, see Equation (III-48), should be performed.

$$-(A_3 n^{(0)} \gamma - n^{(0)3} \gamma^3) 2 \tilde{\pi}^{(1)}(y - \gamma_0^{(0)}) \left(\frac{\partial}{\partial y} \tilde{\pi}^{(1)}(y - \gamma_0^{(0)}) \right) \int_{y - \gamma_0^{(0)}}^y \tilde{\pi}^{(1)}(y') \partial y' =$$

$$\frac{1}{2} (n^{(0)3} \gamma^3 - A_3 n^{(0)} \gamma) (l^{(2m)} g^{(m)} \tilde{\chi}^{(1)3} - l^{(2m)} g^{(m)*} \tilde{\chi}^{(1)2} \tilde{\chi}^{(1)*})$$

(III-55)

$$-n^{(0)2} \gamma^2 \left(\frac{\partial}{\partial y} \tilde{\pi}^{(2)}(y - \gamma_0^{(0)}) \right) \int_{y - \gamma_0^{(0)}}^y \tilde{\pi}^{(1)}(y') \partial y' = -n^{(0)2} \gamma^2 \left\{ \left(\sum_{q=0}^{\infty} l^{(2m)} \left(\tilde{\chi}_{(2m, 2q, \gamma)}^{(2)} + \tilde{\chi}_{(2m, 0, \gamma)}^{(2)} \right) \right) (\tilde{\chi}^{(1)} g^{(m)} - \tilde{\chi}^{(1)} g^{(m)*}) + \frac{1}{2} A_5 (l^{(2m)} g^{(m)} \tilde{\chi}^{(1)3} - l^{(2m)} g^{(m)*} \tilde{\chi}^{(1)2} \tilde{\chi}^{(1)*}) \right\}$$

(III-56)

$$-A_3 n^{(0)} \gamma \frac{\partial}{\partial y} \tilde{\pi}^{(1)}(y - \gamma_0^{(0)}) \int_{y - \gamma_0^{(0)}}^y \tilde{\pi}^{(1)2}(y') \partial y' = -A_3 n^{(0)} \gamma \frac{1}{4} \left(\frac{1}{2} l^{(m)} g^{(2m)} \tilde{\chi}^{(1)3} + (2i l^{(m)} \gamma_0^{(0)} - \frac{1}{2} l^{(m)*} g^{(2m)}) \tilde{\chi}^{(1)2} \tilde{\chi}^{(1)*} \right)$$

(III-57)

$$-(n^{(0)} \gamma)^2 \frac{\partial}{\partial y} \tilde{\pi}^{(1)}(y - \gamma_0^{(0)}) \int_{y - \gamma_0^{(0)}}^y \tilde{\pi}^{(2)}(y') \partial y' = -n^{(0)2} \gamma^2 \left\{ (i \tilde{\chi}^{(1)} l^{(m)} - i \tilde{\chi}^{(1)*} l^{(m)*}) \left(\sum_{q=0}^{\infty} \left(\tilde{\chi}_{(2m, 2q, \gamma)}^{(2)} + \tilde{\chi}_{(2m, 0, \gamma)}^{(2)} \right) \frac{-i}{2} g^{(2m)} + \left(\tilde{\chi}_{(0, 2q, \gamma)}^{(2)} + \tilde{\chi}_{(0, 0, \gamma)}^{(2)} \right) \gamma_0^{(0)} \right) \right. \\ \left. + A_5 \frac{1}{4} \left(\frac{1}{2} l^{(m)} g^{(2m)} \tilde{\chi}^{(1)3} + (2i l^{(m)} \gamma_0^{(0)} - \frac{1}{2} l^{(m)*} g^{(2m)}) \tilde{\chi}^{(1)2} \tilde{\chi}^{(1)*} \right) \right\}$$

(III-58) #

From the analysis of the combustion chamber flow it can be shown that

$\tilde{\chi}_{(0, 2, 0)}^{(2)} = \tilde{\chi}_{(2, 2, 0)}^{(2)} = 0$. This follows from the fact that the Dini Expansions in terms of J_2 , which appear in the inhomogeneous parts of the second order equations, contain no constant terms.

$$-A_3 n^{(1)} r \tilde{\pi}^{(1)}(y - Y_0^{(1)}) [\tilde{\pi}^{(1)2}] = -\frac{1}{4} n^{(1)} r (l^{(1)} g^{(2m)} \tilde{\chi}^{(1)3} + l^{(1)*} g^{(2m)*} \tilde{\chi}^{(1)2} \tilde{\chi}^{(1)*}) \quad (\text{III-59})$$

$$\begin{aligned} -n^{(1)2} r^2 \tilde{\pi}^{(1)}(y - Y_0^{(1)}) [\tilde{\pi}^{(2)}] = & -n^{(1)2} r^2 \left\{ \frac{1}{2} (\tilde{\chi}^{(1)} l^{(m)} + \tilde{\chi}^{(1)*} l^{(m)*}) \sum_{q=0}^{\infty} g^{(2m)} (\tilde{\chi}_{(2m,q)}^{(2)} \right. \\ & \left. + \tilde{\chi}_{(2m,q)}^{(2)}) + \frac{1}{4} A_5 (l^{(m)} g^{(2m)} \tilde{\chi}^{(1)3} + l^{(m)*} g^{(2m)*} \tilde{\chi}^{(1)2} \tilde{\chi}^{(1)*}) \right\} \quad (\text{III-60}) \end{aligned}$$

$$\begin{aligned} -\frac{1}{2} n^{(1)3} r^3 \frac{\partial^2 \tilde{\pi}^{(1)}(y - Y_0^{(1)})}{\partial y^2} \left(\int_{y - Y_0^{(1)}}^y \tilde{\pi}^{(1)}(y') \partial y' \right)^2 = & -\frac{1}{8} n^{(1)3} r^3 (l^{(m)} g^{(m)2} \tilde{\chi}^{(1)3} \\ & + (l^{(m)*} g^{(m)2} - 2 l^{(m)} g^{(m)*} g^{(m)}) \tilde{\chi}^{(1)2} \tilde{\chi}^{(1)*}) \quad (\text{III-61}) \end{aligned}$$

$$Y_0^{(2)} n^{(1)} r \frac{\partial}{\partial y} \tilde{\pi}^{(1)}(y - Y_0^{(1)}) = Y_0^{(2)} n^{(1)} r i l^{(m)} \tilde{\chi}^{(1)} \quad (\text{III-62})$$

$$n^{(2)} r [\tilde{\pi}^{(1)}] = n^{(2)} r g^{(m)} \tilde{\chi}^{(1)} \quad (\text{III-63})$$

Using Expressions (III-49) through (III-63) (to evaluate $\hat{I}^{(3)}$) and to appropriate first and second order relations, then the transformation into complex variables and the eigenfunction expansion of Equation (III-47) gives: #

Note that q' is used as the index in the expansion of the third order quantities while q (without a prime) was used in the second order expansion.

$$\begin{aligned}
 & \sum_{m_3, n_3, q'} \left\{ (1 - \bar{q}^2 \mathcal{W}^{(m_3)}) \frac{\partial}{\partial z} \tilde{F}_{(m_3, n_3, q')}^{(3)} - i m_3 \omega^{(3)} \mathcal{W}^{(m_3)} \tilde{F}_{(m_3, n_3, q')}^{(3)} \right\} \\
 &= \sum_{q'=1}^{\infty} \left\{ (Q_{(3m, 3\nu, q')}^{(3)} e^{3im\gamma} + Q_{(m, 3\nu, q')}^{(3)} e^{im\gamma}) \cos 3\nu\theta J_{3\nu}(S_{(3\nu, q')}) \sqrt{\frac{\Psi}{\Psi_w}} \right. \\
 & \quad \left. + (Q_{(3m, \nu, q')}^{(3)} e^{3im\gamma} + Q_{(m, \nu, q')}^{(3)} e^{im\gamma}) \cos \nu\theta J_{\nu}(S_{(\nu, q')}) \sqrt{\frac{\Psi}{\Psi_w}} \right\} \\
 & \quad + \left\{ (n^{(1)} \gamma i \gamma_0^{(2)} e^{(m)} + n^{(2)} \gamma g^{(m)}) \tilde{R}^{(1)} + i \omega^{(2)} \mathcal{W}^{(m)} \tilde{\Phi}^{(1)} \right\} \cos \nu\theta J_{\nu}(S_{(\nu, q')}) \sqrt{\frac{\Psi}{\Psi_w}}
 \end{aligned}$$

(III-64)

which must be satisfied by the standing-wave solution.

The $Q_{(m_3, n_3, q')}^{(3)}$ terms appearing in the above equation are defined as follows:

$$\begin{aligned}
 Q_{(3m, 3\nu, q')}^{(3)} &= \left\{ N_{(3\nu, q')}^{(3)} \left(\frac{1}{4} A_6 - \frac{1}{48} A_7 - \frac{1}{16} \mathcal{W}^{(3m)} A_8 \right) \tilde{R}^{(3)} + \sum_{q'=1}^{\infty} \left\{ \frac{1}{4} A_{(3\nu, q')}^{(2)} \left(\right. \right. \right. \\
 & \quad \tilde{R}_{(2m, 2\nu, q')}^{(2)} \tilde{R}^{(1)} A_{14} - \tilde{R}^{(1)} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, q')}^{(2)} - \tilde{R}_{(2m, 2\nu, q')}^{(2)} \frac{d}{dz} \tilde{\Phi}^{(1)} + \bar{q}^2 \left(\frac{d}{dz} \tilde{\Phi}^{(1)} \right) \left(\frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, q')}^{(2)} \right) \mathcal{W}^{(3m)} \left. \right. \left. \right\} \\
 & \quad + \mathcal{W}^{(3m)} \frac{1}{2} \bar{q} \tilde{\Phi}^{(1)} \tilde{\Phi}_{(2m, 2\nu, q')}^{(2)} \frac{1}{\Psi_w} (B_{(3\nu, q')}^{(2)} - \frac{1}{2} C_{(3\nu, q')}^{(2)}) \left. \right\}
 \end{aligned}$$

(III-65) #

In this and the following equations $\tilde{R}_{(m, n, q)}^{(2)}(\mathbf{z})$ describes the \mathbf{z} dependence of the second-order density. The subscripts (m, n, q) indicate the transverse dependence of the particular term (i.e., eigenfunction).

$$\begin{aligned}
 Q_{(m, 2\nu, g)}^{(3)} = & \left\{ N_{(2\nu, g)}^{(4)} \left(\frac{1}{4} A_{12} - \frac{1}{16} (A_7 + 3 A_8 \mathcal{W}^{(m)}) \right) \tilde{R}^{(1)2} \tilde{R}^{(1)*} \right. \\
 & + \sum_{q=1}^{\infty} \left\{ A_{(2\nu, g)}^{(2q)} \left(\frac{1}{4} A_{15} \tilde{R}_{(2m, 2\nu, g)}^{(2)} \tilde{R}^{(1)} + \frac{1}{4} A_{16} \tilde{R}^{(1)} (\tilde{R}_{(0, 2\nu, g)}^{(2)} \right. \right. \\
 & + \tilde{R}_{(0, 2\nu, g)}^{(2)*}) - \frac{1}{4} (\tilde{R}^{(1)*} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)} + \tilde{R}^{(1)} (\frac{d}{dz} \tilde{\Phi}_{(0, 2\nu, g)}^{(2)} + \\
 & \frac{d}{dz} \tilde{\Phi}_{(0, 2\nu, g)}^{(2)*}) + \frac{d}{dz} \tilde{\Phi}^{(1)*} \tilde{R}_{(2m, 2\nu, g)}^{(2)} + \frac{d}{dz} \tilde{\Phi}^{(1)} (\tilde{R}_{(0, 2\nu, g)}^{(2)} + \tilde{R}_{(0, 2\nu, g)}^{(2)*}) \\
 & - \bar{q}^2 (\frac{d}{dz} \tilde{\Phi}^{(1)*} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)*} + \frac{d}{dz} \tilde{\Phi}^{(1)} (\frac{d}{dz} \tilde{\Phi}_{(0, 2\nu, g)}^{(2)} + \frac{d}{dz} \tilde{\Phi}_{(0, 2\nu, g)}^{(2)*}) \mathcal{W}^{(m)}) \\
 & + \frac{1}{\psi_w} \bar{q} \mathcal{W}^{(m)} \left((\frac{1}{2} B_{(2\nu, g)}^{(2q)} - \frac{1}{4} C_{(2\nu, g)}^{(2q)}) (\tilde{\Phi}^{(1)*} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)} \right. \\
 & \left. + \tilde{\Phi}^{(1)} (\tilde{\Phi}_{(0, 2\nu, g)}^{(2)} + \tilde{\Phi}_{(0, 2\nu, g)}^{(2)*}) \right) \left. \right\} \quad (III-66)
 \end{aligned}$$

$$\begin{aligned}
 Q_{(3m, \nu, g)}^{(3)} = & \left\{ N_{(\nu, g)}^{(4)} A_{13} \tilde{R}^{(1)3} + \sum_{q=0}^{\infty} \left\{ \frac{1}{4} A_{(\nu, g)}^{(2q)} (A_{14} \tilde{R}^{(1)} \tilde{R}_{(2m, 2\nu, g)}^{(2)} \right. \right. \\
 & - \tilde{R}^{(1)} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)} + \mathcal{W}^{(3m)} \bar{q}^2 \frac{d}{dz} \tilde{\Phi}^{(1)} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)} - \frac{d}{dz} \tilde{\Phi}^{(1)} \tilde{R}_{(2m, 2\nu, g)}^{(2)}) \\
 & + \frac{1}{2} A_{(\nu, g)}^{(0)} (A_{14} \tilde{R}^{(1)} \tilde{R}_{(2m, 0, g)}^{(2)} - \frac{d}{dz} \tilde{\Phi}^{(1)} \tilde{R}_{(2m, 0, g)}^{(2)} - \tilde{R}^{(1)} \frac{d}{dz} \tilde{\Phi}_{(2m, 0, g)}^{(2)} \\
 & + \bar{q}^2 \mathcal{W}^{(3m)} \frac{d}{dz} \tilde{\Phi}^{(1)} \frac{d}{dz} \tilde{\Phi}_{(2m, 0, g)}^{(2)}) + (\frac{1}{2} B_{(\nu, g)}^{(2q)} + \frac{1}{4} C_{(\nu, g)}^{(2q)}) \bar{q} \mathcal{W}^{(3m)} \frac{1}{\psi_w} \tilde{\Phi}^{(1)} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)} \\
 & \left. + B_{(\nu, g)}^{(0)} \bar{q} \mathcal{W}^{(3m)} \tilde{\Phi}^{(1)} \tilde{\Phi}_{(2m, 0, g)}^{(2)} \right\} \quad (III-67)
 \end{aligned}$$

$$\begin{aligned}
Q_{(m, \nu, g)}^{(2)} = & \left\{ N_{(\nu, g)}^{(4)} A_{17} \tilde{R}^{(1)*} \tilde{R}^{(1)*} + \sum_{g=0}^{\infty} \left\{ \frac{1}{4} A_{(\nu, g)}^{(2g)} \left(A_{15} R_{(2m, 2\nu, g)}^{(2)} R^{(1)*} \right. \right. \right. \\
& + A_{16} \tilde{R}^{(1)} \left(\tilde{R}_{(0, 2\nu, g)}^{(2)} + \tilde{R}_{(0, 2\nu, g)}^{(2)*} \right) + \psi^{(m)} \bar{q}^2 \left(\frac{d}{dz} \tilde{\Phi}^{(1)*} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)} \right. \\
& + \frac{d}{dz} \tilde{\Phi}^{(1)} \left(\frac{d}{dz} \tilde{\Phi}_{(0, 2\nu, g)}^{(2)} + \frac{d}{dz} \tilde{\Phi}_{(0, 2\nu, g)}^{(2)*} \right) \left. \right) - \tilde{R}^{(1)*} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)} - \tilde{R}^{(1)} \left(\frac{d}{dz} \tilde{\Phi}_{(0, 2\nu, g)}^{(2)} + \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)*} \right) \\
& - \tilde{R}_{(2m, 2\nu, g)}^{(2)} \frac{d}{dz} \tilde{\Phi}^{(1)} - \frac{d}{dz} \tilde{\Phi}^{(1)} \left(\tilde{R}_{(0, 2\nu, g)}^{(2)} + \tilde{R}_{(0, 2\nu, g)}^{(2)*} \right) + \frac{1}{2} A_{(\nu, g)}^{(0, g)} \left(\tilde{R}_{(2m, 0, g)}^{(2)} R^{(1)*} A_{15} \right. \\
& + A_{16} \tilde{R}^{(1)} \left(\tilde{R}_{(0, 0, g)}^{(2)} + \tilde{R}_{(0, 0, g)}^{(2)*} \right) + \psi^{(m)} \bar{q}^2 \left(\frac{d}{dz} \tilde{\Phi}^{(1)*} \frac{d}{dz} \tilde{\Phi}_{(2m, 0, g)}^{(2)} \right. \\
& + \frac{d}{dz} \tilde{\Phi}^{(1)} \left(\frac{d}{dz} \tilde{\Phi}_{(0, 0, g)}^{(2)} + \frac{d}{dz} \tilde{\Phi}_{(0, 0, g)}^{(2)*} \right) \left. \right) - R^{(1)*} \frac{d}{dz} \tilde{\Phi}_{(2m, 0, g)}^{(2)} - \tilde{R}^{(1)} \left(\frac{d}{dz} \tilde{\Phi}_{(0, 0, g)}^{(2)} \right. \\
& + \frac{d}{dz} \tilde{\Phi}_{(0, 0, g)}^{(2)*} \left. \right) - \tilde{R}_{(2m, 0, g)}^{(2)} \frac{d}{dz} \tilde{\Phi}^{(1)} - \frac{d}{dz} \tilde{\Phi}^{(1)} \left(\tilde{R}_{(0, 0, g)}^{(2)} + \tilde{R}_{(0, 0, g)}^{(2)*} \right) \left. \right\} \\
& + \frac{1}{\psi_w} \left(\frac{1}{2} B_{(\nu, g)}^{(2g)} + \frac{1}{4} C_{(\nu, g)}^{(2g)} \right) \psi^{(m)} \bar{q} \left(\tilde{\Phi}^{(1)*} \tilde{\Phi}_{(2m, 2\nu, g)}^{(2)} + \tilde{\Phi}^{(1)} \left(\tilde{\Phi}_{(0, 2\nu, g)}^{(2)} \right. \right. \\
& + \tilde{\Phi}_{(0, 2\nu, g)}^{(2)*} \left. \right) + \bar{q} \psi^{(m)} \frac{1}{\psi_w} B_{(\nu, g)}^{(0, g)} \left(\tilde{\Phi}^{(1)} \tilde{\Phi}_{(2m, 0, g)}^{(2)} + \tilde{\Phi}^{(1)} \left(\tilde{\Phi}_{(0, 0, g)}^{(2)} + \tilde{\Phi}_{(0, 0, g)}^{(2)*} \right) \right) \left. \right\}
\end{aligned}$$

where

$$\begin{aligned}
 A_6 = & \frac{1}{4}(A_1 A_5 + A_2) g^{(3m)} - \frac{1}{4} A_4 \ell^{(2m)} g^{(m)} - \frac{1}{4} n^{(m)2} r^2 A_5 \ell^{(2m)} g^{(m)} - \frac{1}{4} n^{(m)3} r^3 \ell^{(m)} g^{(m)2} \\
 & + \frac{1}{2} (n^{(m)3} r^3 - A_3 n^{(m)} r) \ell^{(2m)} g^{(m)} - \frac{1}{2} n^{(m)2} r^2 A_5 \ell^{(2m)} g^{(m)} - \frac{1}{8} (A_2 n^{(m)} r \ell^{(m)} g^{(2m)} + A_5 n^{(m)2} r^2 \ell^{(m)} g^{(2m)} \\
 & + n^{(m)3} r^3 \ell^{(m)} g^{(m)2}) - \frac{1}{4} \ell^{(m)} g^{(2m)} r n^{(m)} (A_3 + n^{(m)} r A_5)
 \end{aligned}
 \tag{III-69}$$

$$A_7 = \frac{5}{2} r - \frac{3}{2} r^2 - 1
 \tag{III-70}$$

$$A_8 = \bar{q}^2 \left(\frac{1}{2} r - \frac{1}{16} r^2 - \frac{1}{3} \right) + \frac{1}{3} (r - 2)
 \tag{III-71}$$

$$A_9 = A_1 g^{(3m)} - n^{(m)2} r^2 (2 \ell^{(2m)} g^{(m)} + \frac{3}{2} \ell^{(m)} g^{(2m)})
 \tag{III-72}$$

$$A_{10} = A_1 g^{(m)} - n^{(m)2} r^2 (g^{(2m)} \ell^{(m)*} - \ell^{(2m)} g^{(m)*} - \frac{1}{2} \ell^{(m)*} g^{(2m)})
 \tag{III-73}$$

$$A_{11} = A_1 g^{(m)} - n^{(m)2} r^2 (g^{(m)} + i \ell^{(m)} Y_0^{(m)})
 \tag{III-74}$$

$$\begin{aligned}
 A_{12} = & \frac{3}{4} g^{(m)} (A_1 A_5 + A_2) - \frac{1}{4} (A_4 (2g^{(m)} + l^{(2m)} g^{(m)*}) + n^{(0)2} \gamma^2 A_5 (2g^{(m)} \\
 & + l^{(2m)} g^{(m)*}) - n^{(0)3} \gamma^3 l^{(m)*} g^{(m)2} + (A_5 n^{(0)2} \gamma^2 + A_3 n^{(0)} \gamma) (2il^{(m)} Y_0^{(0)} \\
 & - \frac{1}{2} l^{(m)*} g^{(2m)}) + A_3 n^{(0)} \gamma l^{(m)*} g^{(2m)} + n^{(0)2} \gamma^2 A_5 l^{(m)*} g^{(2m)}) - \frac{1}{2} (n^{(0)3} \gamma^3 \\
 & - A_3 n^{(0)} \gamma - A_5 n^{(0)2} \gamma^2) l^{(2m)} g^{(m)*} - \frac{1}{8} n^{(0)3} \gamma^3 (l^{(m)*} g^{(m)2} \\
 & - 2l^{(m)} g^{(m)} g^{(m)*})
 \end{aligned}$$

(III-75)

$$A_{13} = \frac{3}{4} A_6 - \frac{1}{16} (A_7 + 3W^{(2m)} A_8)$$

(III-76)

$$A_{14} = A_9 + \gamma - 1 - W^{(2m)}$$

(III-77)

$$A_{15} = A_{10} + \gamma - 1 - W^{(m)}$$

(III-78)

$$A_{16} = A_{11} + \gamma - 1 - W^{(m)}$$

(III-79)

$$A_{17} = \frac{3}{4} A_{12} - \frac{3}{16} (A_7 + 3W^{(m)} A_8)$$

(III-80)

The coefficients $A_{(3\nu, q')}^{(j, q)}$, $A_{(\nu, q')}^{(j, q)}$, $B_{(3\nu, q')}^{(j, q)}$, $B_{(\nu, q')}^{(j, q)}$, $C_{(3\nu, q')}^{(j, q)}$, $C_{(\nu, q')}^{(j, q)}$ and $N_{(\nu, q')}^{(\nu, \nu)}$ (for $j = 0, 2$) are defined in Appendix C.

The corresponding third-order boundary condition to be satisfied by the travelling wave solution is:

$$\sum_{m_3, n_3, q'} \left\{ (1 - \bar{q}^2 \mathbb{W}^{(m_3)}) \frac{\partial}{\partial z} \tilde{F}_{(m_3, n_3, q')}^{(3)} - i m_3 \omega^{(n)} \mathbb{W}^{(m_3)} \tilde{F}_{(m_3, n_3, q')}^{(3)} \right\}$$

$$= \sum_{q'=1}^{\infty} \left\{ Q_{(3m, 3\nu, q')}^{(3)} e^{3i(\nu\theta + y)} J_{3\nu}(S_{(3\nu, q')} \sqrt{\frac{\Psi}{\Psi_w}}) \right.$$

$$+ Q_{(m, \nu, q')}^{(3)} e^{i(\nu\theta + y)} J_{\nu}(S_{(\nu, q')} \sqrt{\frac{\Psi}{\Psi_w}})$$

(III-81)

where

$$Q_{(3m, 3\nu, q')}^{(3)} = 2 N_{(3\nu, q')}^{(4N)} A_{18} + 2 Q_{(3m, 3\nu, q')}^{(3)} S$$

(III-82)

and

$$Q_{(m, \nu, q')}^{(3)} = \frac{4}{3} N_{(\nu, q')}^{(4N)} A_{17} \tilde{R}^{(1)2} \tilde{R}^{(1)*} + \sum_{q=0}^{\infty} \left\{ \frac{1}{2} (A_{(\nu, q')}^{(2S)} A_{15} \tilde{R}^{(1)*} \tilde{R}_{(2m, 2\nu, q)}^{(2)} \right.$$

$$+ A_{(\nu, q')}^{(0, \eta)} \tilde{R}^{(1)} (\tilde{R}_{(0, 0, q)}^{(2)} + \tilde{R}_{(0, 0, q)}^{(2)*}) A_{16} + \mathbb{W}^{(m)} \bar{q}^2 (A_{(\nu, q')}^{(2S)} \frac{d}{dz} \tilde{\Phi}^{(1)*} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, q)}^{(2)} \\$$

$$+ A_{(\nu, q')}^{(0, \eta)} \frac{d}{dz} \tilde{\Phi}^{(1)} (\frac{d}{dz} \tilde{\Phi}_{(0, 0, q)}^{(2)} + \frac{d}{dz} \tilde{\Phi}_{(0, 0, q)}^{(2)*}) - A_{(\nu, q')}^{(2S)} R^{(1)*} \frac{d}{dz} \tilde{\Phi}_{(2m, 2\nu, q)}^{(2)} - A_{(\nu, q')}^{(0, \eta)} R^{(1)} (\frac{d}{dz} \tilde{\Phi}_{(0, 0, q)}^{(2)} \\$$

$$+ \frac{d}{dz} \tilde{\Phi}_{(0, 0, q)}^{(2)*}) - A_{(\nu, q')}^{(2S)} \tilde{R}_{(2m, 2\nu, q)}^{(2)} \frac{d}{dz} \tilde{\Phi}^{(1)*} - A_{(\nu, q')}^{(0, \eta)} \frac{d}{dz} \tilde{\Phi}^{(1)} (\tilde{R}_{(0, 0, q)}^{(2)} + \tilde{R}_{(0, 0, q)}^{(2)*}) \\$$

$$+ \bar{q} \mathbb{W}^{(m)} \frac{1}{\Psi_w} ((B_{(\nu, q')}^{(2S)} - \frac{1}{2} C_{(\nu, q')}^{(2S)}) \tilde{\Phi}_{(2m, 2\nu, q)}^{(2)} \tilde{\Phi}^{(1)*} + B_{(\nu, q')}^{(0, \eta)} \tilde{\Phi}^{(1)} (\tilde{\Phi}_{(0, 0, q)}^{(2)} \\$$

$$+ \tilde{\Phi}_{(0, 0, q)}^{(2)*}))$$

(III-83)

$$A_{18} = \frac{1}{4} A_6 - \frac{1}{48} A_7 - \frac{1}{16} A_8 \psi^{(3m)} \quad (\text{III-84})$$

After separation of variables, each of the third-order eigenfunction coefficients which appear in the eigenfunction expansion of $\tilde{F}^{(3)}$ will have to satisfy a boundary condition of the following form:

$$(1 - \bar{q}^2 \psi^{(m)}) \frac{d}{dz} \tilde{\Phi}_{(m, n, q)}^{(3)} - i m_3 \omega^{(m)} \psi^{(m)} \tilde{\Phi}_{(m, n, q)}^{(3)} = Q_{(m, n, q)}^{(3)} \quad (\text{III-85}) \quad \#$$

When the third-order harmonic with subscripts $(m_3, n_3, q') = (1, 1, 1)$ is analyzed, the following expression should be added to the inhomogeneous part of Equation (III-85):

$$(i n^{(m)} Y_0^{(2)} e^{(m)} \gamma + n^{(2)} \gamma g^{(m)}) \tilde{R}'' + i \omega^{(2)} \psi^{(m)} \tilde{\Phi}'' \quad (\text{III-86})$$

The analysis presented in this chapter shows that one of the harmonics in the series expansion of each of the higher order boundary conditions is proportional to the eigenfunction in first-order solution. Combining these higher-order harmonics with the first-order boundary condition yields:

$$\begin{aligned} R^{(m)} \{ \epsilon \tilde{\Phi}'' + \epsilon^2 \tilde{\Phi}_{(m, n, q)}^{(2)} + \epsilon^3 \tilde{\Phi}_{(m, n, q)}^{(3)} \} &= \epsilon^2 \{ (i Y_0^{(2)} e^{(m)} n^{(m)} + n^{(2)} g^{(m)}) \gamma \tilde{R}'' \\ &+ i \omega^{(m)} \psi^{(m)} \tilde{\Phi}'' \} + \epsilon^3 \{ Q_{(m, n, q)}^{(3)} + (i Y_0^{(2)} e^{(m)} n^{(m)} + n^{(2)} g^{(m)}) \gamma \tilde{R}'' \\ &+ i \omega^{(2)} \psi^{(m)} \tilde{\Phi}'' \} \end{aligned} \quad (\text{III-87})$$

The subscripts T or S were omitted from Equation (III-85) as the same form of boundary condition should be satisfied by the travelling as well as standing wave solutions.

where

$$R^{(m)} = \left(\left(1 - \bar{g}^2 W^{(m)} \right) \frac{d}{dz} - im\omega^{(n)} W^{(m)} \right) \quad (\text{III-88})$$

is a linear operator. As will be seen in a later chapter Equation (III-87) will be a key factor in the determination of the relationship between the eigenvalues of the problem.

When one dimensional oscillations are analyzed, the appropriate combustion zone boundary conditions are given directly by Expressions (III-30), (III-32) and (III-45) which are respectively the first, second and third order boundary conditions. In that case, eigenfunction expansions are not necessary. Using the proper transformation these expressions can be shown to be identical to the boundary conditions derived by Sirignano in his study of nonlinear longitudinal instability.

The limiting process used in the derivation of Equation (III-3) led to a nonlinear transverse boundary condition which is locally equivalent to the one-dimensional nonlinear boundary condition.

In conclusion, the boundary conditions derived in this section represent the interaction between the combustion processes and the fluid mechanical processes and can be applied to the study of the behavior and stability of three-dimensional, finite-amplitude, periodic pressure waves.

CHAPTER IV.

ANALYSIS OF COMBUSTION CHAMBER FLOW

Introduction

In this chapter the solutions of the equations describing the behavior of finite-amplitude, three-dimensional, periodic pressure-waves, that frequently occur in the combustion chamber of liquid-propellant rocket engines, will be obtained. As has already been noted, a specific combustion model assuming the existence of a concentrated combustion zone at the injector-face[#] will be considered. Assuming that there is no interaction between the oscillations inside the combustion chamber and the feed-system, the motor under consideration consists of an infinitesimally thin combustion zone located at the injector-end of a cylindrical combustion chamber which connects to a converging-diverging nozzle (see Figure 6).

During steady operation hot combustion products, which are generated inside the concentrated combustion zone, pass through the combustion chamber at constant speed and then enter the nozzle where they are accelerated to supersonic speeds before leaving the engine. However, the operating conditions inside the rocket engine may be unstable to a small disturbance, in the case of linearly unstable system, or to the sudden introduction of a finite-amplitude disturbance. In this situation the flow conditions inside the engine are characterized by a pattern of three-dimensional waves that are super-imposed upon the steady flow. Unsteady operation is also characterized by an oscillatory combustion process which, under proper feedback conditions, can provide the necessary amount of energy which is needed, for maintaining the wave-motion inside the engine. The understanding of this feedback mechanism (or mechanisms) and the structure of the wave-pattern inside the engine are some of the objectives of this work. In the present study these objectives will be pursued by obtaining the solutions of the equations describing the fully-developed wave-motion inside a cylindrical chamber and requiring the solutions to satisfy some specified boundary-conditions repre-

[#] For a discussion regarding the validity of this assumption see footnote on page 107. One could have considered instead the distributed combustion case. However, because of the complexity of the equations describing the nonlinear effects (i.e., the second and third order equations) and for the sake of clarity it was decided to study the "simpler" case presented in this chapter.

senting the effects of the combustion process on one end and the nozzle on the other[#] (see Figure 6).

In the absence of any dissipative effects (i.e., viscosity and heat conduction), the two boundary-conditions, which are imposed at the two ends of the combustion-chamber, are the only means of transferring energy into or out of the system. When the amount of energy input is equal to the amount of energy output the system is said to be neutrally stable and the amplitudes of the oscillations taking place inside the chamber will neither grow or decay. These oscillations will stay neutral as long as the energy balance is not disturbed. In the analogous problem in acoustics, where there is no mass generation and no steady-state through flow, the above boundary conditions are replaced by the classical solid-wall boundary-conditions which require the vanishing of the axial components of velocity at the two ends of the cylindrical chamber. Consequently no energy transfer can take place across these boundaries and the oscillations which occur inside the chamber can continue indefinitely.

In addition to simplifying the Equations describing the combustion chamber flow, the use of a concentrated combustion zone model would enable us to study the effect of high Mach number, at the exit of the combustion-chamber, upon the stability of the system as well as the stability of three-dimensional mixed modes. None of these effects has previously been investigated analytically in the case of three-dimensional oscillations.^{##}

It has been stated previously¹ that in the case of high frequency pressure oscillations the location of the combustion zone can have a profound effect upon the stability of the system. Letting all (or most of) the combustion take place in the vicinity of a pressure antinode of a given mode, where the interaction between the pressure waves and the combustion process

[#] It was pointed out in Chapter II that in analyzing problems of this sort it is more convenient to separate the combustion chamber from the nozzle and obtain their solutions separately. The effect of the nozzle, upon the combustion chamber flow, is then introduced through a specially-derived boundary condition best known as Nozzle Admittance Relation, that must be satisfied by the solutions of the equations describing the flow conditions inside the combustion chamber.

^{##} Reardon² and Scala¹² have studied the stability of small amplitude transverse oscillations in the distributed combustion case. The complexity of the equations describing this phenomenon forced them, however, to limit their investigations to flows with low Mach numbers at the combustion chamber exit and to the analysis of the purely transverse mode of oscillation only.

is the greatest, would represent the worse possible location as far as the stability of the combustion process is concerned. It will be shown later, by use of the combustion zone boundary-condition, that the magnitude of the axial component of the velocity, at the injector face, is of the order of the local Mach number. Consequently the injector face is expected to be (at least in the case of low Mach number flow) in the vicinity of a pressure antinode and the specific example which is analyzed in this chapter represents a very unstable engine.[#]

Use of the concentrated combustion model results in the elimination of the mass, momentum and energy sources (or sinks) from the equations describing the flow conditions inside the combustion chamber. This results in the elimination of a well-known stabilizing effect: namely, the drag between liquid drops and hot gases.

Because of the complicated form of the general Nozzle Admittance Relations (in the case of finite-amplitude oscillations) and the length of computer time necessary for the numerical computation of all the quantities which appear in the coefficients of the general Admittance Relation, the flow inside the combustion-chamber will be assumed to be irrotational. In this case the hot gases leaving the combustion zone must be irrotational and have constant entropy (i.e., the non-steady entropy perturbations must be zero).

We shall now proceed with the analysis of the problem. As was explained in the last section of Chapter II, the equations describing the nozzle flow can also be used in the present analysis and consequently no additional equations will have to be derived. As a matter of fact the restriction that nozzle walls be slowly convergent that had to be imposed, in the case of the nozzle flow, in order to use the approximation $\delta n \sim dr$ (see Equation (II-34) and the paragraph preceding it) is no longer necessary. Since the combustion chamber walls are parallel to the axis of symmetry, the relation $\delta n = dr$ holds exactly throughout the combustion chamber and no approximations are necessary.

Using the boundary conditions which were derived in the last two

[#] This condition is true for the mixed and longitudinal modes but may not always be true for the purely transverse mode.

chapters we shall now proceed with the determination of the eigenvalues (which in the present problem are included in the combustion zone boundary condition) and the solution of the equations which describe the combustion-chamber flow. The latter will be obtained by following the same procedure as the one used in the solution of the equations describing the nozzle flow. In solving the equations describing the nozzle flow it is convenient to let the initial point, $\phi = 0$, be at the nozzle throat, where the equations are singular, and then proceed with the numerical integration in the direction of the combustion-chamber (ϕ is taken to be negative in this direction). In the present analysis the initial point is chosen to coincide with the injector face where all of the combustion is assumed to take place. In order to distinguish between the equations describing the nozzle flow and those describing the combustion-chamber flow we shall let, in the latter case, the independent variable z replace the variable ϕ (although both have still the same definition) in describing the axial dependence of the solution. The points $z = 0$ and $z = z_e$ represent respectively the injector face and the nozzle entrance.

First Order Flow

The equations and solutions describing the first-order flow in the combustion chamber can be obtained by simply substituting Equations (II-204) and (II-205) (which implies $\frac{d\bar{q}}{dz} = 0$) into the corresponding expressions which appear in the first-order analysis of the nozzle flow. In particular, the substitution of these equations into Equation (II-91) and use of the relation

$$\psi_w = \frac{1}{2} \bar{q} \quad (\text{IV-1})$$

give

$$\bar{q}^2 (\bar{c}^2 - \bar{q}^2) \frac{d^2}{dz^2} \tilde{\Phi}^{(w)} - \bar{q}^2 2m\omega^{(w)} \frac{d}{dz} \tilde{\Phi}^{(w)} + (m^2 \omega^{(w)2} - S_{(v,h)}^2) \tilde{\Phi}^{(w)} = 0 \quad (\text{IV-2})^\#$$

Since $\sigma^{(1)}$ and $C_1^{(1)}$ (which in this case represent respectively the

[#] A wiggle is used here in order to distinguish between quantities describing the combustion chamber flow and the corresponding quantities which describe the nozzle flow (and have no "wiggle" mark).

first order perturbation of the entropy and vorticity evaluated at the concentrated combustion zone) are identically zero under the assumption of irrotationality.

It is interesting to note that to first order the combustion chamber flow can be irrotational and still have non-zero entropy perturbation. This result follows directly from Equations (II-66) and (II-67) which give the expressions for the transverse components of the vorticity. The latter contain terms which are proportional to the product of the entropy perturbation and the gradient of steady-state velocity distribution (i.e., $\frac{d\bar{u}}{dz}$) which is taken to be identically zero in our particular example.

Equation (IV-2) is a second order, homogeneous ordinary differential equation with constant coefficients whose solution must satisfy the following first-order boundary conditions

$$(1 - \bar{q}^2 \bar{W}''') \frac{d}{dz} \tilde{\Phi}'' - i m \omega''' \bar{W}''' \tilde{\Phi}'' = 0$$

(IV-3)

at the injector face (where $z = 0$) and

$$\frac{U'''}{V'''} = \bar{\mu}$$

(IV-4) ^{#, ##}

at the nozzle entrance (where $z = z_e$). These boundary conditions were derived in the previous chapters. Using Equations (II-95a) and (II-95b) Equation (IV-4) can be rewritten in the following equivalent form:

$$\frac{d}{dz} \tilde{\Phi}'' - \bar{\mu} \tilde{\Phi}'' = 0$$

(IV-4a)

A more general form of this boundary condition (which also applies to higher order flows) is given in Equation (II-189). In first order analysis the quantity $\bar{\mu}$ is identically zero. The quantity $\bar{\mu}$ depends on the particular nozzle under consideration and can be obtained from the numerical integration of Equation (II-192).

For definition of $\bar{\mu}$ see Equation (IV-9a).

As can be verified by direct substitution, the solution of Equation (IV-2) can be written in the following form

$$\tilde{\Phi}^{(1)} = C_2^{(1)} e^{r_1^{(1)} z} + C_3^{(1)} e^{r_2^{(1)} z} \quad (\text{IV-5})$$

where $C_2^{(1)}$ and $C_3^{(1)}$ are some arbitrary complex constants and

$$r_{1,2}^{(1)} = \frac{i m \omega^{(1)}}{1 - \bar{q}^2} \pm \frac{S_{(1,h)} \sqrt{1 - \bar{q}^2 - \left(\frac{m \omega^{(1)}}{S_{(1,h)}} \right)^2}}{\bar{q} (1 - \bar{q}^2)} \quad (\text{IV-6})$$

Letting

$$\mathcal{Q}^{(m)} = 1 - \bar{q}^2 \mathcal{W}^{(m)} = \mathcal{Q}^{(m)}(n^{(m)}, \gamma^{(m)}, \omega^{(m)}, \bar{q} \dots) \quad (\text{IV-7})$$

and substituting Equation (IV-5) into Equation (IV-3) gives

$$\frac{C_3^{(1)}}{C_2^{(1)}} = \frac{i \omega^{(1)} m \mathcal{W}^{(1)} - \mathcal{Q}^{(1)} r_1^{(1)}}{\mathcal{Q}^{(1)} r_2^{(1)} - i m \omega^{(1)} \mathcal{W}^{(1)}} \quad (\text{IV-8})$$

Assuming that $\bar{\mu}$ is known, the substitution of Equation (IV-5) into Equation (IV-4a) gives:

$$\frac{C_3^{(1)}}{C_2^{(1)}} = \frac{r_1^{(1)} - \bar{\mu}}{\bar{\mu} - r_2^{(1)}} e^{(r_1^{(1)} - r_2^{(1)}) z_e} = W \quad (\text{IV-9})$$

where

$$\bar{\mu} = \mathcal{F}\mu = \left(\frac{A_{c.c.}}{A_{throat}} \frac{1}{1 + \frac{r-1}{2} \bar{q}_f^2} \right)^{1/2} \mu \quad (\text{IV-9a})^\#$$

Equating Equations (IV-8) and (IV-9) and using Equations (III-36), and (IV-7) gives

$$n'' \gamma (1 - e^{-i\omega'' \tau''}) = 1 - H(\omega'', \bar{q}_f, S_{\omega, n}, \dots) \quad (\text{IV-10})$$

where

$$H = \frac{W r_2'' + r_1''}{im\omega'' + \bar{q}_f^2 r_1'' + W(im\omega'' + \bar{q}_f^2 r_2'')} \quad (\text{IV-11})$$

In its given form Equation (IV-10), which is also known as the characteristic equation, exhibits the energy balance which exists at the stability limits (to first order or for linear analysis). The energy supplied by the combustion process, which is represented by the left-hand side of Equation (IV-10) must be absorbed by the fluid mechanical processes which take place inside the combustion chamber and the nozzle and which are represented by the right-hand side of Equation (IV-10). The fact that Equation (IV-10) is complex implies that there is a phase condition, as well

[#] In this equation \mathcal{F} is a proportionality constant that must be introduced to compensate for the fact that the nondimensionalization scheme used in the solution of the combustion chamber flow is different from the one used in the solution of the nozzle flow.

as an amplitude condition relating the rates of energy release and energy absorption for neutral oscillations.

Separation of Equation (IV-10) into its real and imaginary parts gives:

$$n^{(0)} \gamma (1 - \cos \omega^{(0)} \tau_0^{(0)}) = 1 - H_r \quad (\text{IV-12a})$$

$$n^{(0)} \gamma \sin \omega^{(0)} \tau_0^{(0)} = -H_i \quad (\text{IV-12b})$$

Considering $\omega^{(0)}$ as a parameter, we can solve the above equations for $n^{(0)}$ and $\tau_0^{(0)}$; getting:

$$n^{(0)}(\omega^{(0)}) = \frac{(H_r - 1)^2 + H_i^2}{2(1 - H_r)\gamma} \quad (\text{IV-13a})$$

$$\tau_0^{(0)}(\omega^{(0)}) = \frac{1}{\omega^{(0)}} \sin^{-1} \left(-\frac{H_i}{n^{(0)} \gamma} \right) \quad (\text{IV-13b})$$

where $\omega^{(0)} \tau_0^{(0)}$ is determined modulo 2π . From Equation (IV-13a) it can be seen that $(1 - H_r) > 0$ is a necessary condition for a physically realizable system which requires $n^{(0)} > 0$. Once μ , \bar{q} , $S(\nu_h)$ and $\omega^{(0)}$ are known, $n^{(0)}$ and $\tau_0^{(0)}$ can be determined from Equations (IV-13a,b). Using these equations and letting $\omega^{(0)}$ vary yields a plot of $n(\omega^{(0)})$ vs. $\tau_0^{(0)}(\omega^{(0)})$ which divides the n, τ plane into stable and unstable regions while the curve itself represents the locus of neutral oscillations.

(See for example Figure 13). This curve which is parabolic in shape contains two values of $\tau_{\cdot}^{(n)}$ (each of which occurs, however, at a different frequency) for each value of $n^{(0)}$. These values of $\tau_{\cdot}^{(n)}$, which can be related to the frequency of the system, represent upper and lower frequency limits above and below which the system operation is stable.[#] Obtaining this plot was the primary objective of most of the early works (see for example References 2 and 12) that were limited to the study of linear combustion instability. In these cases if, for a given engine, some average values of its time-lag and interaction index were known, then using such stability plots it would have been possible to determine whether the engine is linearly stable or unstable to a small disturbance of a particular frequency.

In the present study where finite-amplitude waves are being considered, the solution of the first-order equations represents the first-term only of the asymptotic series which approximates the solution of the problem.

Assuming that $c_2^{(1)}$ has been "absorbed" by ϵ , then the first order solution can be written in the following form:

$$\epsilon \tilde{\Phi}^{(n)} = \epsilon (e^{r_1^{(n)} z} + W e^{r_2^{(n)} z}) \quad (\text{IV-14})$$

where W is determined in Equation (IV-9).

Substitution of $\tilde{\Phi}^{(n)}$ (as given in Equation (IV-14)) into Equations (II-95a) through (II-95d) yields the following expressions

$$\epsilon \tilde{U}^{(n)} = \epsilon (r_1^{(n)} e^{r_1^{(n)} z} + W r_2^{(n)} e^{r_2^{(n)} z}) \quad (\text{IV-15a})$$

$$\begin{aligned} \epsilon \tilde{P}^{(n)} = & -\epsilon \left(\bar{q}^2 \frac{d}{dz} \tilde{\Phi}^{(n)} + i m \omega^{(n)} \tilde{\Phi}^{(n)} \right) = -\epsilon \left((i m \omega^{(n)} + \bar{q}^2 r_1^{(n)}) e^{r_1^{(n)} z} \right. \\ & \left. + (i m \omega^{(n)} + \bar{q}^2 r_2^{(n)}) W e^{r_2^{(n)} z} \right) \end{aligned} \quad (\text{IV-15b})$$

$$\epsilon \tilde{R}^{(n)} = \epsilon \tilde{P}^{(n)} \quad (\text{IV-15c})$$

[#] This statement holds for a given $n^{(n)}$.

and

$$\tilde{V}'' = \tilde{\Phi}'' = \tilde{W}''$$

(IV-15d)

which respectively describe the axial dependence of the first order solutions for the axial component of the velocity, pressure, density and the transverse components of the velocity.

To obtain the complete solutions of the first order equations the above expressions must be substituted into Equation (II-84).

In order to proceed with the solution of the higher order equations, the complete solution of the first order nozzle flow will be necessary. The quantity μ which was necessary for the calculation of W can be obtained by numerical integration of Equation (II-192) which starts at the nozzle throat and proceeds in the direction of the combustion chamber. Once μ is available, the first-order solution of the nozzle flow can be obtained from Equation (II-194). Since in reality the combustion chamber continuously and smoothly transforms itself into a nozzle, the solution Φ'' is expected to be continuous across the nozzle entrance. Consequently equating Equations (IV-14) and (II-194) gives:

$$\Phi_n''(0) e^{\int_0^{\varphi_e} \mu(\varphi) d\varphi'} = (e^{r_1'' z_e} + W e^{r_2'' z_e}) \mathcal{F}$$

(IV-16)[#]

Separating Equation (IV-16) into its real and imaginary parts results in two real equations which can be used in the determination of the real and imaginary parts of $\Phi_n''(0)$. Once Φ'' is known throughout the combustion-chamber and the subsonic portion of the nozzle, we can proceed with the solution of the higher order equations.

Using the method of separation of variables the solution of the first-order equations describing the travelling-wave motion, inside the combustion

In the actual calculations the value of φ_e is determined by the Mach number which, at this point, must equal the local Mach number of the combustion chamber flow. See Equation (IV-9a) for definition of \mathcal{F} .

chamber and the nozzle, can be obtained in exactly the same manner as the solution of the equations describing the standing wave motion. Since both cases yield identical results, the analysis of the equations describing the travelling wave motion will not be given here. The complete first-order solutions, which describe the travelling-wave motion (which can rotate either in the clockwise or counter clockwise directions, can be obtained by substituting Equations (IV-15a) through (IV-15d) into Equations (II-84) through (II-87).

Higher-Order Effects in the Combustion Chamber Flow

To study the behavior of finite amplitude waves, the expressions representing the coefficients of the higher powers of ϵ in the asymptotic representation of the dependent variables and the eigenvalues, which are given in Equations (II-14) and (III-26), must be determined. A calculation of a sufficient number of these coefficients is expected to provide us with an approximate description of the pressure wave form and the expressions representing the eigenvalue perturbations. The relationship between the eigenvalue perturbations is expected to predict where in the vicinity of the neutral stability curve (which has been determined in the first-order analysis) in an n vs. $\tau^{(n)}$ plot finite amplitude oscillations (which may be stable or unstable) can occur. The amplitude of these oscillations is expected to depend on the normal distance between the location of the oscillations and the neutral stability curve; and to increase with this distance.

In the limit as this distance goes to zero, the location of these oscillations (which will then have an infinitesimal amplitude) will coincide with one of the points along the neutral curve. #

The next step in the analysis will be the solution of the second order equations. Substitution of Equations (II-204) and (II-205) into Equations (II-36) through (II-41) (for $j=2$) and use of the appropriate form of the first-order solutions (i.e., standing or travelling-wave solutions) give the proper form of the second order equations describing the combustion chamber flow. As in the case of the nozzle flow these equations are not separable and the method of eigenfunction expansion must be used in their solution. Applying this method of solution to the set of equations which resulted

See Appendix A of Reference 4 for a mechanical analogy of this phenomenon.

from the above simplifications and following the same procedure as in the solution of the second order nozzle flow lead to the derivation of the following inhomogeneous ordinary differential equation for each of the coefficients $\tilde{\Phi}_{(km,nv,q)}^{(2)}$, which appear in the eigenfunction expansion of $\tilde{F}^{(2)}$:

$$\bar{q}^2 (1 - \bar{q}^2) \frac{d^2}{d\varphi^2} \tilde{\Phi}_{(km,nv,q)}^{(2)} - 2ikm\omega^{(1)} \frac{d}{d\varphi} \tilde{\Phi}_{(km,nv,q)}^{(2)} + (k^2 m^2 \omega^{(1)2} - S_{(nv,q)}^2) \tilde{\Phi}_{(km,nv,q)}^{(2)} = \tilde{I}_{(km,nv,q)}^{(2)}(\varphi)$$

(IV-17)^{#,###}

In the case of standing wave motion there are four equations of the above form for each value of the summation index q which is used in the eigenfunction expansion (see for example the expansions given in Equations (II-108) through (II-123)). The only exception to the above statement is the case $q = 0$ where there are only two such ordinary differential equations and $\tilde{\Phi}_{(km,nv,q)}^{(2)} \neq 0$ only for $n = 0$.^{##} For all other values of q the subscripts (km,nv,q) can be equal to any one of the following four combinations: $(2m,2v,q)$, $(2m,o,q)$, $(o,2v,q)$ and (o,o,q) . When the flow under consideration is irrotational the inhomogeneous part of Equation (IV-17) which corresponds to each of the above combination of subscripts can be expressed in

In its given form this equation describes both the standing and travelling wave motion. The particular case being considered is determined by the form of $\tilde{I}_{(km,nv,q)}^{(2)}(\varphi)$.

This is due to the fact that Dini-Expansions in terms of J_0 (which appear in the inhomogeneous part of the second order equations) are the only ones which have a constant term which corresponds to the first root (which is zero) of the equation $\frac{d}{dx} J_0(x) = 0$, present in their expansions.

Note that in the case of longitudinal oscillations $\tilde{I}_{(km,nv,q)}^{(2)}(\varphi)$ contains terms which are solutions of the homogeneous part of this equation. The analysis of this problem will result in the appearance of secular solutions.

the following form:

$$\tilde{I}_{(2m, \lambda, \nu, q)}^{(2)} = \frac{1}{2} \left(A_{(\lambda, \nu, q)} \tilde{X} + \frac{1}{\psi_w} (B_{(\lambda, \nu, q)} - \nu^2 C_{(\lambda, \nu, q)}) \tilde{\lambda} \right) \quad (\text{IV-18a})$$

$$\tilde{I}_{(2m, 0, q)}^{(2)} = \frac{1}{2} \left(A_{(0, q)} \tilde{X} + \frac{1}{\psi_w} (B_{(0, q)} + \nu^2 C_{(0, q)}) \tilde{\lambda} \right) \quad (\text{IV-18b})$$

$$\tilde{I}_{(0, \lambda, \nu, q)}^{(2)} = \frac{1}{2} \left(A_{(\lambda, \nu, q)} \tilde{\underline{1}} + \frac{1}{\psi_w} (B_{(\lambda, \nu, q)} - \nu^2 C_{(\lambda, \nu, q)}) \tilde{\underline{T}} \right) \quad (\text{IV-18c})$$

$$\tilde{I}_{(0, 0, q)}^{(2)} = \frac{1}{2} \left(A_{(0, q)} \tilde{\underline{1}} + \frac{1}{\psi_w} (B_{(0, q)} + \nu^2 C_{(0, q)}) \tilde{\underline{T}} \right) \quad (\text{IV-18d})$$

where

$$\begin{aligned} \tilde{X} = \frac{1}{2} \{ & \tilde{R}^{(4)} (i\omega^{(4)} \tilde{R}^{(4)} + \bar{q}^2 \tilde{R}^{(4)'}) - \bar{q}^2 \tilde{U}^{(4)} \tilde{R}^{(4)'} - \bar{q}^2 \left(\frac{2-r}{2} 2\tilde{R}^{(4)'} \tilde{R}^{(4)'} \right. \\ & \left. - \bar{q}^2 \tilde{U}^{(4)} \tilde{U}^{(4)'} \right) - 2i\omega^{(4)} \left(\frac{2-r}{2} \tilde{R}^{(4)2} - \frac{1}{2} \bar{q}^2 \tilde{U}^{(4)2} \right) \} \end{aligned} \quad (\text{IV-19})$$

$$\begin{aligned} \tilde{\underline{1}} = \frac{1}{2} \{ & \tilde{R}^{(4)*} \tilde{R}^{(4)'} \frac{r}{2} \bar{q}^2 + i\omega^{(4)} \tilde{R}^{(4)*} \tilde{R}^{(4)} - \bar{q}^2 \tilde{U}^{(4)*} \tilde{R}^{(4)'} - \bar{q}^2 \frac{2-r}{2} \tilde{R}^{(4)} \tilde{R}^{(4)'} \\ & + \frac{1}{2} \bar{q}^4 (\tilde{U}^{(4)*} \tilde{U}^{(4)} + \tilde{U}^{(4)} \tilde{U}^{(4)'}) \} \end{aligned} \quad (\text{IV-20})$$

$$\tilde{\lambda} = \bar{q} \left(\bar{q}^2 \tilde{v}^{(1)} \tilde{v}^{(1)'} + i \omega^{(1)} \tilde{v}^{(1)2} - \tilde{v}^{(1)} \tilde{p}^{(1)} \right)$$

(IV-21)

$$\tilde{T} = \frac{1}{2} \bar{q}^3 \left(\tilde{v}^{(1)} \tilde{v}^{(1)*'} + \tilde{v}^{(1)'} \tilde{v}^{(1)*} \right) - \bar{q} \tilde{v}^{(1)} \tilde{p}^{(1)*}$$

(IV-22)

The above equations could also be obtained by direct substitution of Equations (II-204) and (II-205) into Equations (II-143) through (II-147) which are the inhomogeneous parts of the second order equations describing the irrotational nozzle flow.

In addition to the above given expressions for $\tilde{I}_{(km,n\nu,q)}^{(2)}$, the inhomogeneous parts and the boundary conditions of the second order equations contain other components which have the same spatial and time dependence as the first order solution. The consequences of the presence of these terms, which also include the first order eigenvalue perturbations, will be discussed in the following section.

To obtain an analytic solution of Equation (IV-17), for all the possible combinations of the subscripts (km,nν,q), its homogeneous part must be rewritten in a more useful form.

Substituting the definitions of $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\lambda}$ and \tilde{T} together with the appropriate form of the first order solutions into Equations (IV-18a) through (IV-18d) gives:

$$\tilde{I}_{(2m,n\nu,q)}^{(2)} = \sum_{i=1}^3 T_{i(2m,n\nu,q)} h_i$$

(IV-23)

$$\tilde{I}_{(0,n\nu,q)}^{(2)} = \sum_{i=1}^4 V_{i(0,n\nu,q)} g_i$$

(IV-24)

where n can be zero or two and

$$T_{i(2m, n\nu, q)} = \frac{1}{2} (A_{(n\nu, q)} \tilde{X}_i + D_{(n\nu, q)} \tilde{\lambda}_i) \quad (\text{IV-25})$$

$$V_{i(o, n\nu, q)} = \frac{1}{2} (A_{(n\nu, q)} \tilde{\Gamma}_i + D_{(n\nu, q)} \tilde{T}_i) \quad (\text{IV-26})$$

$$D_{(o, q)} = \frac{1}{\psi_w} (B_{(o, q)} + \nu^2 C_{(o, q)})$$

$$D_{(2, q)} = \frac{1}{\psi_w} (B_{(2, q)} - \nu^2 C_{(2, q)}) \quad (\text{IV-27})$$

$$h_1 = \tilde{\Phi}_1^{(1)2} = e^{2r_1'' z} = e^{\tilde{h}_1 z} \quad (\text{IV-28a})$$

$$h_2 = \tilde{\Phi}_1^{(1)} \tilde{\Phi}_2^{(1)} = e^{(r_1'' + r_2'') z} = e^{\tilde{h}_2 z}$$

$$h_3 = \tilde{\Phi}_2^{(1)2} = e^{2r_2'' z} = e^{\tilde{h}_3 z} \quad (\text{IV-28b})$$

$$g_1 = \tilde{\Phi}_1^{(1)} \tilde{\Phi}_2^{(1)} = e^{(r_1'' + r_1^{(1)*}) z} = e^{\tilde{g}_1 z} \quad (\text{IV-28c})$$

$$g_1 = \tilde{\Phi}_1^{(1)} \tilde{\Phi}_2^{(1)} = e^{(r_1'' + r_1^{(1)*}) z} = e^{\tilde{g}_1 z} \quad (\text{IV-29a})$$

$$g_2 = \tilde{\Phi}_1^{(\omega)} \tilde{\Phi}_2^{(\omega)*} = e^{(r_1^{(\omega)} + r_2^{(\omega)*})z} = e^{\tilde{g}_2 z}$$

(IV-29b)

$$g_3 = \tilde{\Phi}_1^{(\omega)*} \tilde{\Phi}_2^{(\omega)} = e^{(r_1^{(\omega)*} + r_2^{(\omega)})z} = e^{\tilde{g}_3 z}$$

(IV-29c)

$$g_4 = \tilde{\Phi}_2^{(\omega)} \tilde{\Phi}_1^{(\omega)*} = e^{(r_2^{(\omega)} + r_1^{(\omega)*})z} = e^{\tilde{g}_4 z}$$

(IV-29d)

$$\begin{aligned} \tilde{X}_1 = \frac{1}{2} \big(& -i\omega^{(\omega)^3}(\delta-1) + 2i\omega^{(\omega)}\bar{q}^4\delta r_1^{(\omega)^2} - 3\omega^{(\omega)^2}\bar{q}^2(\delta-1)r_1^{(\omega)} \\ & + i\omega^{(\omega)}\bar{q}^4(\delta-1)r_1^{(\omega)^2} + \bar{q}^4(\bar{q}^2(\delta-1)-2)r_1^{(\omega)^3} \big) \end{aligned}$$

(IV-30a)

$$\begin{aligned} \tilde{X}_2 = \frac{1}{2} W \big(& -i\omega^{(\omega)^3}(\delta-1)2 + 4i\omega^{(\omega)}\bar{q}^4\delta r_1^{(\omega)}r_2^{(\omega)} - 3\omega^{(\omega)^2}\bar{q}^2(\delta-1)(r_1^{(\omega)} + r_2^{(\omega)}) \\ & + i\omega^{(\omega)}\bar{q}^4(\delta-1)(r_1^{(\omega)^2} + r_2^{(\omega)^2}) + \bar{q}^4(\bar{q}^2(\delta-1)+2)(r_1^{(\omega)^2}r_2^{(\omega)} + r_2^{(\omega)^2}r_1^{(\omega)}) \big) \end{aligned}$$

(IV-30b)

$$\begin{aligned} \tilde{X}_3 = \frac{1}{2} W^2 \big(& -i\omega^{(\omega)^3}(\delta-1) + 2i\omega^{(\omega)}\bar{q}^4\delta r_2^{(\omega)^2} - 3\omega^{(\omega)^2}\bar{q}^2(\delta-1)r_2^{(\omega)} + i\omega^{(\omega)}\bar{q}^4(\delta-1)r_2^{(\omega)^2} \\ & + \bar{q}^4(\bar{q}^2(\delta-1)+2)r_2^{(\omega)^3} \big) \end{aligned}$$

(IV-30c)

$$\begin{aligned} \tilde{Z}_1 = \frac{1}{2} \big(& i\omega^{(\omega)^3} - \omega^{(\omega)^2}\bar{q}^2\left(\frac{\delta-1}{2}\right)r_1^{(\omega)*} + \omega^{(\omega)^2}\bar{q}^2\left(\frac{\delta+2}{2}\right)r_1^{(\omega)} + i\omega^{(\omega)}\bar{q}^4(2\bar{q}^2+1)r_1^{(\omega)}r_1^{(\omega)*} \\ & - i\omega^{(\omega)}\bar{q}^4\frac{\delta}{2}r_1^{(\omega)^2} - i\omega^{(\omega)}\bar{q}^4\frac{2-\delta}{2}r_1^{(\omega)*} + \bar{q}^4\frac{\delta+3}{2}r_1^{(\omega)^2}r_1^{(\omega)*} - \bar{q}^4\frac{\bar{q}^2(2-\delta)-1}{2}r_1^{(\omega)*2}r_1^{(\omega)} \big) \end{aligned}$$

(IV-31a)

$$\begin{aligned} \tilde{I}_2 = \frac{1}{2} W & \left(i\omega^{(0)3} - \omega^{(0)2} \bar{q}^2 \frac{4-\delta}{2} r_1^{(1)*} + \omega^{(0)2} \bar{q}^2 \frac{\delta+2}{2} r_2^{(1)} + i\omega^{(0)} \bar{q}^{-2} (2\bar{q}^2+1) r_2^{(1)} r_1^{(1)*} \right. \\ & - i\omega^{(0)} \bar{q}^{-4} \frac{\delta}{2} r_2^{(1)2} - i\omega^{(0)} \bar{q}^{-2} \frac{2-\delta}{2} r_1^{(1)*2} + \bar{q}^4 \frac{\delta+3}{2} r_2^{(1)2} r_1^{(1)*} \\ & \left. - \bar{q}^4 (\bar{q}^2(2-\delta)-1) r_1^{(1)*2} r_1^{(1)} \right) \end{aligned} \quad (\text{IV-31b})$$

$$\begin{aligned} \tilde{I}_3 = \frac{1}{2} W^* & \left(i\omega^{(0)3} - \omega^{(0)2} \bar{q}^2 \frac{4-\delta}{2} r_2^{(1)*} + \omega^{(0)2} \bar{q}^{-2} \frac{\delta+2}{2} r_1^{(1)} + i\omega^{(0)} \bar{q}^{-2} (2\bar{q}^2+1) r_1^{(1)} r_2^{(1)*} \right. \\ & - i\omega^{(0)} \bar{q}^{-4} \frac{\delta}{2} r_1^{(1)2} - i\omega^{(0)} \bar{q}^{-2} \frac{2-\delta}{2} r_2^{(1)*2} + \bar{q}^4 \frac{\delta+3}{2} r_1^{(1)2} r_2^{(1)*} \\ & \left. - \bar{q}^2 (\bar{q}^2(2-\delta)-1) r_2^{(1)*2} r_1^{(1)} \right) \end{aligned} \quad (\text{IV-31c})$$

$$\begin{aligned} \tilde{I}_4 = \frac{1}{2} W W^* & \left(i\omega^{(0)3} - \omega^{(0)2} \bar{q}^2 \frac{4-\delta}{2} r_2^{(1)*} + \omega^{(0)2} \bar{q}^{-2} \frac{\delta+2}{2} r_2^{(1)} + i\omega^{(0)} \bar{q}^{-2} (2\bar{q}^2+1) r_2^{(1)} r_2^{(1)*} \right. \\ & - i\omega^{(0)} \bar{q}^{-4} \frac{\delta}{2} r_2^{(1)2} - i\omega^{(0)} \bar{q}^{-2} \frac{2-\delta}{2} r_2^{(1)*2} + \bar{q}^4 \frac{\delta+3}{2} r_2^{(1)2} r_2^{(1)*} \\ & \left. - \bar{q}^4 (\bar{q}^2(\delta-1)-1) r_2^{(1)*2} r_2^{(1)} \right) \end{aligned} \quad (\text{IV-31d})$$

$$\tilde{\lambda}_1 = 2\bar{q} (\bar{q}^2 r_1^{(1)} + i\omega^{(0)}) \quad (\text{IV-32a})$$

$$\tilde{\lambda}_2 = 2\bar{q} W (\bar{q}^2 (r_1^{(1)} + r_2^{(1)}) + 2i\omega^{(0)}) \quad (\text{IV-32b})$$

$$\tilde{\lambda}_3 = 2\bar{q} (\bar{q}^2 r_2^{(1)} + i\omega^{(0)}) \quad (\text{IV-32c})$$

$$\tilde{T}_1 = \bar{q} \left(\frac{3}{2} \bar{q}^2 r_1^{(1)*} + \frac{1}{2} \bar{q}^2 r_1^{(1)} + i\omega^{(0)} \right) \quad (\text{IV-33a})$$

$$\tilde{T}_2 = \bar{q} W \left(\frac{3}{2} \bar{q}^2 r_1^{(1)*} + \frac{1}{2} \bar{q}^2 r_2^{(1)} + i \omega^{(1)} \right) \quad (\text{IV-33b})$$

$$\tilde{T}_3 = \bar{q} W^* \left(\frac{3}{2} \bar{q}^2 r_2^{(1)*} + \frac{1}{2} \bar{q}^2 r_1^{(1)} + i \omega^{(1)} \right) \quad (\text{IV-33c})$$

and

$$\tilde{T}_4 = \bar{q} W W^* \left(\frac{3}{2} \bar{q}^2 r_2^{(1)*} + \frac{1}{2} \bar{q}^2 r_1^{(1)} + i \omega^{(1)} \right) \quad (\text{IV-33d})$$

Since the inhomogeneous part of Equation (IV-17) can be written as a summation of exponentials, none of which is a solution of the homogeneous part of the equation, its solution can easily be obtained by use of the method of undetermined coefficients (see Reference 13). Using this method the general solution of Equation (IV-17), which is the sum of the homogeneous and particular solutions, for the case when $(km, n\nu, q) = (2m, n\nu, q)$ and $n=0,2$ is found to be:

$$\tilde{\Phi}_{(2m, n\nu, q)}^{(2)} = C_{2(2m, n\nu, q)}^{(2)} e^{r_{1(2m, n\nu, q)}^{(2)} z} + C_{3(2m, n\nu, q)}^{(2)} e^{r_{2(2m, n\nu, q)}^{(2)} z} + \sum_{i=1}^3 \tilde{X}_{i(2m, n\nu, q)} h_i \quad (\text{IV-34})$$

where

$$\tilde{X}_{i(2m, n\nu, q)} = \frac{T_{i(2m, n\nu, q)}}{\bar{q}^2 (1 - \bar{q}^2) \tilde{h}_i^2 - 2 i k m \omega^{(1)} \tilde{h}_i + k^2 m^2 \omega^{(1)2} S_{(n\nu, q)}^2} \quad (\text{IV-35})$$

and when $(km, n\nu, q) = (0, n\nu, q)$ for $n = 0, 2$ the solution of Equation (IV-17) is

$$\tilde{\Phi}_{(0, n\nu, q)}^{(2)} = C_{2(0, n\nu, q)}^{(2)} e^{r_{1(0, n\nu, q)}^{(2)} z} + C_{3(0, n\nu, q)}^{(2)} e^{r_{2(0, n\nu, q)}^{(2)} z} + \sum_{i=1}^4 \tilde{X}_{i(0, n\nu, q)} g_i \quad (\text{IV-36})$$

where now

$$\tilde{X}_{i(0,n\nu,q)} = \frac{V_{i(0,n\nu,q)}}{\bar{q}^2(1-\bar{q}^2)\tilde{g}_i^2 - S_{(n\nu,q)}} \quad (\text{IV-37})$$

In the above equations, $r_1^{(2)}(km, n\nu, q)$ and $r_2^{(2)}(km, n\nu, q)$ are the roots of the following quadratic equation

$$\bar{q}^2(1-\bar{q}^2)r^2 - 2ikm\omega^{(n)}\bar{q}^2r + km^2\omega^{(n)2} - S_{(n\nu,q)}^2 = 0$$

which resulted from the substitution of $\tilde{\Phi}_h = e^{r_2 z}$ into the homogeneous part of Equation (IV-17); \tilde{h}_i and \tilde{g}_i are defined in Equations (IV-28a) through (IV-29d) and $C_2^{(2)}(km, n\nu, q)$, and $C_3^{(2)}(km, n\nu, q)$ are arbitrary constants of integration that will be determined through the application of the proper boundary conditions at the two ends of the combustion chamber.

In the previous two chapters the second and third order boundary conditions, describing the effects of the concentrated combustion at one end of the combustion chamber and the nozzle at the other, were expressed as eigenfunction expansions containing the same eigenfunctions as the corresponding expansions which describe the combustion chamber flow. Consequently each of the coefficients, $\tilde{\Phi}_{(km, n\nu, q)}^{(j)}$, where $j = 2, 3, \#$ of any one of the eigenfunctions which appear in the series solution of the higher order equations must satisfy two boundary conditions. These conditions pertain to the coefficients of the same eigenfunction which also appears in the series representation of higher order boundary conditions. In the present investigation these boundary conditions can be obtained from Equations (III-43), (II-131), (II-140), (II-141), (II-189), and (IV-9a) which require that the following relations

$$(1 - \bar{q}^2 \omega^{(km)}) \frac{d}{dz} \tilde{\Phi}_{(km, n\nu, q)}^{(2)} - ikm\omega^{(n)} \tilde{\Phi}_{(km, n\nu, q)}^{(2)} = Q_{(km, n\nu, q)}^{(2)} \quad (\text{IV-38})$$

In the present discussion, we let $j = 2, 3$ since the same remarks also apply to the determination of the coefficients of the third order equations.

and

$$\frac{d}{dz} \tilde{\Phi}_{(km, n, q)}^{(2)} - \bar{\mu}_{(km, n, q)} \tilde{\Phi}_{(km, n, q)}^{(2)} = -\bar{c}^2 f_0^{(km)} \Gamma_{(km, n, q)}^{(2)}$$

(IV-39)

be respectively satisfied at the injector face (where $z = 0$) and the nozzle entrance (where $z = z_e$).

Letting

$$A_{(km, n, q)}^{(2)} = \frac{Q_{(km, n, q)}^{(2)} - (1 - \bar{q}^2 \mathbb{W}^{(km)}) \frac{d}{dz} \tilde{\Phi}_{P(km, n, q)}^{(2)} + i k_m \omega^{(0)} \mathbb{W}^{(km)} \tilde{\Phi}_{P(km, n, q)}^{(2)}}{r_{(km, n, q)}^{(2)} (1 - \bar{q}^2 \mathbb{W}^{(km)}) - i k_m \omega^{(0)} \mathbb{W}^{(km)}}$$

(IV-40a)

and

$$B_{(km, n, q)}^{(2)} = \frac{i k_m \omega^{(0)} \mathbb{W}^{(km)} - 1 + \bar{q}^2 \mathbb{W}^{(km)}}{r_{(km, n, q)}^{(2)} (1 - \bar{q}^2 \mathbb{W}^{(km)}) - i k_m \omega^{(0)} \mathbb{W}^{(km)}}$$

(IV-40b)

where $\tilde{\Phi}_{p(km, n, q)}^{(2)}$ is the particular solution of Equation (IV-17), and requiring $\tilde{\Phi}_{(km, n, q)}^{(2)}$ to satisfy the boundary conditions given in Equations (IV-38) and (IV-39) results in a set of two algebraic equations which when solved for the constants of integration give:

$$C_3^{(2)} = \frac{-\bar{c}^2 f_0^{(km)} \Gamma_n^{(2)} - (r_1^{(2)} - \bar{\mu}) e^{r_1^{(2)} z_e^{(2)}} A - \frac{d}{dz} \tilde{\Phi}_{P(z_e)}^{(2)} + \bar{\mu} \tilde{\Phi}_{P(z_e)}^{(2)}}{B^{(2)} e^{r_1^{(2)} z_e^{(2)}} (r_1^{(2)} - \bar{\mu}) + e^{r_2^{(2)} z_e^{(2)}} (r_2^{(2)} - \bar{\mu})}$$

(IV-41a)

$$C_2^{(2)} = A^{(2)} + C_3^{(2)} B^{(2)}$$

(IV-41b)[#]

Once $\tilde{\Phi}_{(km, n, q)}^{(2)}$ is known, the use of Equations (II-137), (II-138), (II-140), (II-141), (II-129), (II-130), (II-131), (II-132), (II-118a,b), (II-204) and (II-205) together with the eigenfunction expansions of $\xi^{(2)}$, $\eta^{(2)}$, $\xi^{(2)}$ and $\pi^{(2)}$, which are given in the second order analysis of the nozzle flow, yield the following series solutions for the second order variables describing the combustion chamber flow:

$$\begin{aligned} \xi_{c.c.}^{(2)} = \sum_{k=0,2} e^{ikmy} \left\{ \sum_{q=1}^{\infty} \frac{d}{dz} \tilde{\Phi}_{(km, q, q)}^{(2)} \cos 2\nu\theta J_{2\nu}(S_{2\nu, q}) \sqrt{\frac{\psi'}{\psi_w}} \right. \\ \left. + \sum_{q=0}^{\infty} \frac{d}{dz} \tilde{\Phi}_{(km, 0, q)}^{(2)} J_0(S_{10, q}) \sqrt{\frac{\psi'}{\psi_w}} \right\} \end{aligned}$$

(IV-42)

$$\begin{aligned} \eta_{c.c.}^{(2)} = \sum_{k=0,2} e^{ikmy} \left\{ \sum_{q=1}^{\infty} \tilde{\Phi}_{(km, q, q)}^{(2)} \cos 2\nu\theta \frac{d}{d\psi} J_{2\nu}(S_{2\nu, q}) \sqrt{\frac{\psi'}{\psi_w}} \right. \\ \left. + \sum_{q=0}^{\infty} \tilde{\Phi}_{(km, 0, q)}^{(2)} \frac{d}{d\psi} J_0(S_{10, q}) \sqrt{\frac{\psi'}{\psi_w}} \right\} \end{aligned}$$

(IV-43)

[#] Because of lack of space the subscripts (km, n, q) have been omitted from Equations (IV-41a) and (IV-41b).

$$\xi_{c.c.}^{(2)} = \sum_{k=0,2} e^{ikmy} \sum_{q=1}^{\infty} \tilde{\Phi}_{(km, n', \frac{q}{f})}^{(2)} \left(\frac{d}{d\theta} \cos 2\nu\theta \right) J_{2\nu}(S_{(2\nu, \frac{q}{f})} \sqrt{\frac{\Psi}{\Psi_w}})$$

(IV-44)

and

$$\begin{aligned} \tilde{\pi}^{(2)} = \sum_{k=0,2} e^{ikmy} \left\{ \sum_{q=1}^{\infty} \tilde{P}_{(km, n', \frac{q}{f})}^{(2)} \cos 2\nu\theta J_{2\nu}(S_{(2\nu, \frac{q}{f})} \sqrt{\frac{\Psi}{\Psi_w}}) \right. \\ \left. + \sum_{q=0}^{\infty} \tilde{P}_{(km, 0, \frac{q}{f})}^{(2)} J_0(S_{(0, \frac{q}{f})} \sqrt{\frac{\Psi}{\Psi_w}}) \right\} \end{aligned}$$

(IV-45)

where

$$\begin{aligned} \tilde{P}_{(2m, n', \frac{q}{f})}^{(2)} = \frac{1}{4} \left(\frac{1}{2} \tilde{R}^{(1)2} - \frac{1}{2} \bar{q}^2 \tilde{U}^{(1)2} \right) A_{(n', \frac{q}{f})} - \frac{1}{4} \bar{q} D_{(n', \frac{q}{f})} \tilde{\Phi}^{(1)2} \\ - (kmi\omega^{(1)} \tilde{\Phi}_{(2m, n', \frac{q}{f})}^{(2)} + \bar{q}^2 \tilde{\Phi}_{(2m, n', \frac{q}{f})}^{(2) \prime}) \end{aligned}$$

(IV-46a)

$$\begin{aligned} \tilde{P}_{(0, n', \frac{q}{f})}^{(2)} = \frac{1}{4} \left(\frac{1}{2} \bar{R}^{(1)} \bar{R}^{(1)*} - \frac{1}{2} \bar{q}^2 \bar{U}^{(1)} \bar{U}^{(1)*} \right) A_{(n', \frac{q}{f})} - \frac{1}{4} \bar{q} D_{(n', \frac{q}{f})} \tilde{\Phi}^{(1)} \tilde{\Phi}^{(1)*} \\ - \bar{q}^2 \tilde{\Phi}_{(0, n', \frac{q}{f})}^{(2) \prime} \end{aligned}$$

(IV-46b)

$n = 0, 2$ and the subscripts c.c indicate that these expressions describe the combustion chamber flow. It should be indicated that the expressions given in the above series solutions are all complex and the calculation of their real parts is necessary for the determination of physically meaningful quantities.

The determination of $\bar{\mu}_{(km,n\nu,q)}$ and $\Gamma_{N(km,n\nu,q)}^{(j)}$ which appear in the Nozzle Admittance Relation, has been discussed in detail in the next to the last section of Chapter II. Once these quantities are known, the function $\bar{\Phi}_{(km,n\nu,q)}^{(2)}$ which describes the nozzle flow can be determined by simply matching the expressions given by Equation (II-202), which still contains an arbitrary constant, and Equations (IV-34) or (IV-36) (depending on the particular eigenfunction which is being considered) at the nozzle entrance. Considering for example Equation (IV-34) and assuming that $C_{2(2m,n\nu,q)}^{(2)}$ and $C_{3(2m,n\nu,q)}^{(2)}$ have already been determined, then applying this matching procedure yields the following result:

$$\begin{aligned} \bar{\Phi}_{(2m,n\nu,q)}^{(2)} = & \mathcal{J} \bar{\Phi}_{(2m,n\nu,q)}^{(2)}(z=z_e) e^{-\int_0^{\phi_e} \mu_{(2m,n\nu,q)}^{(\phi')} d\phi'} \\ & + \int_0^{\phi_e} \bar{C}^2 f_0^{(2m)} \Gamma_{N(2m,n\nu,q)}^{(2)} e^{-\int_0^{\phi'} \mu_{(2m,n\nu,q)}^{(\phi'')} d\phi''} d\phi' \end{aligned}$$

(IV-47)

where \mathcal{J} is a proportionality constant which was defined in Equation (IV-9a).

Following the same procedure as the one used in the analysis of the nozzle flow (see discussion on pages 52-54) the equations and solutions derived so far in this section can be readily modified to describe the second order travelling wave-motion inside the combustion chamber. The details of this straight forward analysis will not be given here.

It will be shown in the next section, by solving for $\tilde{\Phi}_{(m,v,h)}^{(2)}$, that the first order eigenvalue perturbations (i.e., $\omega^{(1)}$, $n^{(1)}$ and $\tau_o^{(1)}$ or $y_o^{(1)}$) are identically zero. In an attempt to find nonzero eigenvalue perturbations the analysis of the problem must be extended to third order. Considering Equations (II-36) through (II-41) (for $j = 3$) as well as the expressions given in the section discussing the third order nozzle flow and repeating the procedure that led to the derivation of Equation (IV-17) results in an identical equation for each of the coefficients which are present in the eigenfunction expansion of $\tilde{F}_{(km,nv,q')}^{(3)}$. In addition each of these coefficients must satisfy two boundary conditions whose general form is identical to the form of the corresponding second order boundary conditions which are given in Equations (IV-38) and (IV-39). The determination of $\tilde{I}_{(km,nv,q')}^{(3)}$, $\tilde{N}_{(km,nv,q')}^{(3)}$ and $Q_{(km,nv,q')}^{(3)}$ requires a complete knowledge of the second order solutions. Since the second order solutions are available in series form, the determination of the exact form of these expressions becomes quite involved. For example the determination of $\tilde{N}_{(km,nv,q')}^{(3)}$ (which is obtained by numerical integration of Equation (II-201)), requires the complete knowledge of the behavior, throughout the nozzle, of each one of the coefficients which appears in the second order eigenfunction expansion of $F^{(2)}$, and consequently results in a considerable numerical effort. It has been estimated that the calculation of $\tilde{N}_{(km,nv,q')}^{(3)}$ will require the simultaneous numerical integration of three to four hundred equations depending on how accurately we expect our solution to be.

Sirignano⁴, in his analysis of finite amplitude longitudinal oscillations, has shown that the third order correction to the pressure wave form is insignificant. In view of the above remarks and since it is reasonable to assume that also in the present analysis (of three-dimensional waves) the third order correction to the pressure wave form is negligible, the following analysis will concentrate on the determination of the eigenvalue perturbations.

For each value of q' , which appears in the summation of the third order eigenfunction expansion, the subscripts $(km, n\nu, q')$ can take on anyone of the following combinations: $(3m, 3\nu, q')$, $(3m, \nu, q')$, $(m, 3\nu, q')$ and (m, ν, q') . In particular whenever $q' = h$ one of these combinations is identical to the one describing the first order solution where $(km, n\nu, q) = (m, \nu, h)$. The discussion of the solution of this component, which is the only one which involves the second order eigenvalue perturbations, will be discussed in the next section.

Determination of the Eigenvalue Perturbations

Review of the analysis that led to the derivation of the higher order Nozzle Admittance Relations and the concentrated combustion zone boundary conditions (both of which are available as eigenfunction expansions) shows that all the eigenvalue perturbations are included in the coefficients of the eigenfunction which has the same time and space dependence as the first order eigenfunction. Thus, the solution describing the behavior of the coefficient of this eigenfunction (to all orders of the analysis) throughout the combustion chamber and which satisfies the appropriate boundary conditions at the two ends of the combustion chamber would yield the correct relationship between the eigenvalue perturbations.

Using Equations (II-204) and (II-205) and repeating the analysis which resulted in the derivation of Equation (II-174) (which applies to the nozzle flow) yields the following result:

$$\begin{aligned} \mathcal{L}_{c.c. (m, \nu, h)} & \left(\epsilon \tilde{\Phi}^{(1)} + \epsilon^2 \tilde{\Phi}_{(m, \nu, h)}^{(2)} + \epsilon^3 \tilde{\Phi}_{(m, \nu, h)}^{(3)} \right) \\ &= \epsilon^2 \left(-2i\omega^{(1)} \tilde{R}^{(1)} + \tilde{I}_{(m, \nu, h)}^{(2)} \right) \\ &+ \epsilon^3 \left(-2i\omega^{(2)} \tilde{R}^{(1)} + \tilde{I}_{(m, \nu, h)}^{(3)} \right) \end{aligned} \quad (IV-48)$$

where $\mathcal{L}_{c.c(m,\nu,h)}$ is a linear operator which represents the operations which are performed on $\tilde{\Phi}_{(km,n\nu,q)}^{(2)}$ on the left-hand-side of Equation (IV-17) and $\tilde{I}_{(m,\nu,h)}^{(j)}$ for $j = 2, 3$ are simplified forms of the analogous expressions which appear in Equation (II-174). The solution of Equation (IV-48) can be written in the following general form

$$\begin{aligned} \tilde{\Phi}_{(m,\nu,h)} = & \epsilon \tilde{\Phi}_h^{(1)} + \epsilon^2 \left(\omega^{(1)} \tilde{\Phi}_{p(m,\nu,h)}^{(2)} + \tilde{\Phi}_{p(m,\nu,h)}^{(2)} \right) \\ & + \epsilon^3 \left(\omega^{(2)} \tilde{\Phi}_{p(m,\nu,h)}^{(3)} + \tilde{\Phi}_{p(m,\nu,h)}^{(3)} \right) \end{aligned} \quad (IV-49)$$

where the subscripts h and p represent respectively homogeneous and particular solutions. As can be seen each of the second and third order particular solutions has been divided into two parts one of which (i.e., $\tilde{\Phi}_{p(m,\nu,h)}^{(j)}$) represents the particular solution which resulted from the presence of terms proportional to $\omega^{(k)}$ ($k = 1, 2$) in the inhomogeneous part of Equation (IV-49) and the other solution (i.e., $\tilde{\Phi}_{p(m,\nu,h)}^{(j)}$) represents the effect of $\tilde{I}_{(m,\nu,h)}^{(j)}$. For the moment it will be convenient to assume that $\tilde{\Phi}_{p(m,\nu,h)}^{(j)}$ (for $j = 2, 3$) is known[#] and only $\tilde{\Phi}_{p(m,\nu,h)}^{(j)}$ will be determined in this section.

Substituting Equations (IV-15b,c) into the inhomogeneous part of Equation (IV-48), noting that the resulting expression contains exponentials which satisfy the homogeneous part of the same equation and using the method of undetermined coefficients yield the following solution for $\tilde{\Phi}_{p(m,\nu,h)}^{(j)}$:

[#] In this case $\tilde{\Phi}_{p(m,\nu,h)}^{(2)} = 0$ (since $\tilde{I}_{(m,\nu,h)}^{(2)} = 0$) and the solution of $\tilde{\Phi}_{p(m,\nu,h)}^{(3)}$ depends on the form of $\tilde{I}_{(m,\nu,h)}^{(3)}$.

$$\begin{aligned} \tilde{\Phi}_{P(m, \nu, h)}^{(j)} &= \frac{i(i m \omega^{(0)} + \bar{q}^2 r_1''')}{\bar{q}^2(1 - \bar{q}^2) r_1''' - i m \omega^{(0)} \bar{q}^2} z e^{r_1''' z} \\ &+ W \frac{i(i m \omega^{(0)} + \bar{q}^2 r_2''')}{\bar{q}^2(1 - \bar{q}^2) r_2''' - i m \omega^{(0)} \bar{q}^2} z e^{r_2''' z} \end{aligned}$$

(IV-49a)

where $j = 2, 3$.

Since it is a solution of a second order ordinary differential equation, $\tilde{\Phi}_{P(m, \nu, h)}^{(j)}$ must satisfy two boundary conditions which are given by Equation (III-87), which applies to the injector face, and (II-189) which applies to the nozzle entrance. In its given form Equation (II-189) applies to any eigenfunction. For the specific case under consideration the following form of the Nozzle Admittance Relation should be used:

$$\begin{aligned} \left(\frac{d}{dz} - \bar{\mu} \right) \left(\epsilon \tilde{\Phi}'' + \epsilon^2 \tilde{\Phi}_{(m, \nu, h)}^{(2)} + \epsilon^3 \tilde{\Phi}_{(m, \nu, h)}^{(3)} \right) \\ = - \bar{C}^2 f_0^{(m)} \left(\epsilon^2 \Gamma_{N(m, \nu, h)}^{(2)} + \epsilon^3 \Gamma_{N(m, \nu, h)}^{(3)} \right) \end{aligned}$$

(IV-50)

Note that the homogeneous solutions of the second and third order equations do not appear explicitly in the solution of $\tilde{\Phi}_{(m, \nu, h)}$ which is given in Equation (IV-49). In this case the latter are assumed to be included in $\tilde{\Phi}_h^{(1)}$. It thus follows that ϵ , the expansion parameter, is the amplitude of $\tilde{\Phi}_h^{(1)}$ which satisfies the homogeneous part of Equation (IV-48) and the homogeneous portions of the boundary conditions which are imposed at the two ends of the combustion chamber. The same result would have been obtained if:

$$\epsilon_1 = \epsilon - (\epsilon^2 + \epsilon^3)$$

was chosen to be the expansion parameter and the homogeneous solutions of the first, second and third order equations (which control the behavior of $\tilde{\Phi}_{(m,\nu,h)}$) were required to satisfy the same homogeneous boundary conditions (which are identical to the first order boundary conditions) at the two ends of the combustion chamber. In this case each of these homogeneous solutions (i.e., $\tilde{\Phi}_h^{(j)}$ for $j = 1, 2, 3$) can be written in the following form.[#]

$$\tilde{\Phi}_{h(m,\nu,h)}^{(j)} = C_2'' (e^{r_1'' z} + W e^{r_2'' z})$$

Letting $C_2^{(1)} = 1$ (which results in no loss of generality of the final result), combining the homogeneous solutions of the first, second and third order equations and replacing ϵ_1^2 and ϵ_1^3 by ϵ^2 and ϵ^3 yield the result given in Equation (IV-49).

As a result of these substitutions, errors of $O(\epsilon^3)$ and $O(\epsilon^4)$ are introduced into the solution of $\tilde{\Phi}_{(m,\nu,h)}$. In the following analysis, it will be shown that the first order eigenvalue perturbations are identically zero. This will result in the elimination of the second order component of $\tilde{\Phi}_{(m,\nu,h)}$. Consequently it becomes convenient to redefine the expansion parameter and let $\epsilon_1 = \epsilon - \epsilon^3$. In this case combining the homogeneous solutions of the first and third order equations and replacing ϵ_1^3 by ϵ^3 introduces errors $O(\epsilon^4)$ or higher. The latter are negligible in the present analysis. To complete the analysis we must now require that the particular solutions of the second and third order equations satisfy the corresponding inhomogeneous boundary conditions at the two ends of the combustion chamber.

The above discussion which resulted in the "elimination" of the homogeneous solutions of the second and third order equations could also be repeated for the corresponding homogeneous solutions (i.e., $\tilde{\Phi}_{h(m,\nu,h)}^{(j)}$)

for $j = 2, 3$ which appear in the analysis of the nozzle flow. Consequently, $\tilde{\Phi}_{(m,\nu,h)}^{(j)}$ and $\Phi_{(m,\nu,h)}^{(j)}$ for $j = 2, 3$ are respectively the particular

[#] The procedure used in obtaining this solution is identical to the one used in the derivation of the first order solution and consequently will not be given here.

solutions of Equations (IV-48) and (II-174). Since $\Phi_{(m,v,h)}^{(j)}$ is continuous across the nozzle entrance $\Phi_{p(m,v,h)}^{(j)}$ must equal $\tilde{\Phi}_{p(m,v,h)}^{(j)}$ at this point. Using the definition of $\Gamma_{N(m,v,h)}^{(j)}$ and the continuity argument given above, it can be shown that $\tilde{\Phi}_{p(m,v,h)}^{(j)}$ for $j = 2, 3$ automatically satisfies the appropriate inhomogeneous higher order nozzle admittance relations which are given in Equation (IV-50). In order to obtain the complete solution of $\tilde{\Phi}_{(m,v,h)}$ (which will lead to the determination of the eigenvalue perturbations) the second and third order inhomogeneous injector end boundary conditions must be satisfied as well.

Substitution of the solution of $\tilde{\Phi}_{(m,v,h)}$ into Equation (III-87) and separation of the resulting expressions according to powers of ϵ yield the following relations

$$R^{(m)}(\tilde{\Phi}_h^{(m)}) = 0 \quad (IV-51a)$$

$$R^{(m)}(\omega^{(j-1)} \tilde{\Phi}_{p(m,v,h)}^{(j)} + \tilde{\Phi}_{p(m,v,h)}^{(j)}) = (i \ell^{(m)} Y_0^{(j-1)} n^{(m)} + n^{(j-1)} g^{(m)}) \tilde{R}^{(m)} + i \omega^{(j-1)} \Pi^{(m)} \tilde{\Phi}^{(m)} + Q_{(m,v,h)}^{(j)} \quad (IV-51b)^{\#}$$

that must be satisfied at $z = 0$. In Equation (IV-51b) $Y_0^{(j-1)}$,

[#] In its given form this relation applies to the cases $j = 2, 3$. It must be remembered, however, that $Q_{(m,v,h)}^{(2)} = 0$.

$n^{(j-1)}$ and $\omega^{(j-1)}$ represent the eigenvalue perturbations $\mathbb{W}^{(m)}$, $\ell^{(m)}$ and $g^{(m)}$ are defined in Equation (III-36) and the operator $\mathcal{R}^{(m)}$ is defined in Equation (III-88). As expected, Equation (IV-51a) is identical with Equation (IV-3) which has been used in the first order analysis and does not have to be considered any more. Rewriting Equation (IV-51b) in a different form yields the following relation to be satisfied by the eigenvalue perturbation.

$$n^{(j-1)} A + Y_0^{(j-1)} B + \omega^{(j-1)} C = Q_1^{(j)} \quad (IV-52)$$

where

$$A = \gamma \tilde{R}^{(0)} g^{(m)} \quad (IV-53a)$$

$$B = i \gamma \ell^{(m)} \tilde{R}^{(0)} \quad (IV-53b)$$

$$C = i \mathbb{W}^{(m)} \tilde{\Phi}^{(0)} + i \omega^{(0)} \mathbb{W}^{(m)} \tilde{\Phi}_{p(m, \gamma, h)}^{(j)} - \mathcal{C}^{(m)} \frac{d}{dz} \tilde{\Phi}_{p(m, \gamma, h)}^{(j)} \quad (IV-53c)$$

$$Q_1 = \mathcal{C}^{(m)} \frac{d}{dz} \Phi_{p(m, \gamma, h)}^{(j)} - i \omega^{(0)} \mathbb{W}^{(m)} \Phi_{p(m, \gamma, h)}^{(j)} - Q_{(m, \gamma, h)}^{(j)} \quad (IV-53d)$$

and $\mathcal{C}^{(m)}$ has been defined in Equation (IV-7).

It will be interesting to investigate the meaning of the expressions presented on the left-hand-side of Equation (IV-52). With this purpose in mind we return to Equation (IV-8) and rewrite it in the following form:

$$f(n, \omega, Y, \dots) = i m \omega \mathbb{W}^{(m)} - \mathcal{C}^{(m)} r_1^{(j)} + W(i m \omega \mathbb{W}^{(m)} - \mathcal{C}^{(m)} r_2^{(j)}) \quad (IV-8a)$$

where W is defined in Equation (IV-9). In Equation (IV-8a) the zeroeth order perturbations of ω , n and Y_0 have been replaced by the corresponding variables which are now functions of ϵ . Using the definitions

of $\tilde{R}^{(1)}$, $\mathbb{W}^{(m)}$, $g^{(m)}$, $\ell^{(m)}$ and $\mathcal{C}^{(m)}$ and differentiating Equation (IV-8a) with respect to n , Y_0 and ω yields the following results:

$$\left(\frac{\partial f}{\partial n}\right)_{\epsilon=0} = g^{(m)} \gamma \tilde{R}_{(0)}^{(1)} \quad (\text{IV-54})$$

$$\left(\frac{\partial f}{\partial Y_0}\right)_{\epsilon=0} = i \gamma n^{(1)} \ell^{(m)} \tilde{R}_{(0)}^{(1)} \quad (\text{IV-55})$$

and

$$\begin{aligned} \left(\frac{\partial f}{\partial \omega}\right)_{\epsilon=0} = & \left(i m \mathbb{W}^{(m)} - \mathcal{C}^{(m)} \frac{\partial r_1^{(1)}}{\partial \omega} \right) + W \left(i m \mathbb{W}^{(m)} - \mathcal{C}^{(m)} \frac{\partial r_2^{(1)}}{\partial \omega} \right) \\ & + \frac{\partial W}{\partial \omega} \left(i m \omega^{(1)} \mathbb{W}^{(m)} - \mathcal{C}^{(m)} r_2^{(1)} \right) \end{aligned} \quad (\text{IV-56})$$

Using the definitions of W and $\bar{\mu}$ (i.e., Equations (IV-9a) and (II-185a)) it can be shown that the last term on the right-hand-side of the above equation is identically zero. Finally using the definitions of $r_1^{(1)}$ and $r_2^{(1)}$ and the solution for $\tilde{\Phi}_{p(m,\nu,h)}^{(j)}$ as given in Equations (IV-6) and (IV-49a) it can be shown that at $z = 0$

$$\frac{\partial r_1^{(1)}}{\partial \omega} + W \frac{\partial r_2^{(1)}}{\partial \omega} = \left(\frac{d}{dz} \tilde{\Phi}_{p(m,\nu,h)}^{(1)} \right)_{z=0} \quad (\text{IV-57})$$

Consideration of the above remarks and the substitution of Equation (IV-57) into Equation (IV-56) yield the following result:

$$\left(\frac{\partial f}{\partial \omega}\right)_{\epsilon=0} = im \mathbb{W}^{(m)} (1 + W) - \mathbb{C}^{(m)} \frac{d}{dz} \tilde{\Phi}_{p(m, \nu, h)}^{(j)} \quad (IV-58)$$

Since $\tilde{\Phi}_{p(m, \nu, h)}^{(j)}$ is identically zero, it immediately follows

that $\left(\frac{\partial f}{\partial n}\right)_{\epsilon=0}$, $\left(\frac{\partial f}{\partial Y_0}\right)_{\epsilon=0}$ and $\left(\frac{\partial f}{\partial \omega}\right)_{\epsilon=0}$ are respectively equal to A, B and C which are defined in Equations (IV-53a), (IV-53b) and (IV-53c). Consequently Equation (IV-52) can be rewritten in the following form:

$$\begin{aligned} A n^{(j-1)} + B Y_0^{(j-1)} + C \omega^{(j-1)} &= \nabla f \cdot \underline{\underline{A}}^{(j-1)} \\ &= \nabla f_r \cdot \underline{\underline{A}}^{(j-1)} + i \nabla f_i \cdot \underline{\underline{A}}^{(j-1)} = Q_i^{(j)} = Q_{ir}^{(j)} + i Q_{ii}^{(j)} \end{aligned} \quad (IV-59)$$

where

$$\underline{\underline{A}}^{(k)} = n^{(k)} \underline{\underline{e}}_n + Y_0^{(k)} \underline{\underline{e}}_{Y_0} + \omega^{(k)} \underline{\underline{e}}_\omega = \underline{\underline{A}}_2^{(k)} + \omega^{(k)} \underline{\underline{e}}_\omega \quad (IV-60)$$

$\underline{\underline{e}}_n$, $\underline{\underline{e}}_{Y_0}$ and $\underline{\underline{e}}_\omega$ are respectively the unit vectors in the n, Y_0 and ω directions and $\underline{\underline{A}}_2^{(k)}$ is the projection of $\underline{\underline{A}}^{(k)}$ on the $\omega = \text{constant}$ plane.

Geometrically[#] ∇f_r and ∇f_i represent respectively the gradients of the three-dimensional surfaces which are determined by the real and

[#] See Figure 7 for illustration of some of the points which will be discussed in the following paragraphs.

imaginary parts of Equation (IV-8a), on page 171, (when $\epsilon = 0$) that is:

$$f_r(n, Y_0, \omega^{(j)} \dots) = 0 \quad (\text{IV-61a})$$

$$f_i(n, Y_0, \omega^{(j)} \dots) = 0 \quad (\text{IV-61b})$$

The intersection of these surfaces in space results in a three-dimensional curve, which will be denoted here by $\tilde{f}(n, Y_0, \omega^{(j)} \dots) = 0$, whose projection on the n, Y_0 plane gives the linear stability plot. At each point along this curve (in space) the vectors ∇f_r and ∇f_i determine a plane which is perpendicular to this curve and consequently contains the vector $\nabla \tilde{f}$ (which represents the gradient of $\tilde{f}(n, Y_0, \omega^{(j)} \dots) = 0$). The latter can be written as a sum of two mutually perpendicular vectors both of which lie in the plane determined by ∇f_r and ∇f_i . These components are chosen in such a way that one of them, say $\nabla \tilde{f}_2$, represents the projection of $\nabla \tilde{f}$ on the plane $\omega_1^{(j)} = \text{constant}$ (where $\omega_1^{(j)}$ represents one of the coordinates of $\tilde{f} = 0$ at the point under consideration). The projection of $\nabla \tilde{f}_2$ on the n, Y_0 plane (which is parallel, of course, to the plane $\omega_1^{(j)} = \text{constant}$) appears as a vector normal to the neutrally stable curve. Similarly, the Displacement Vector $\underline{A}^{(j-1)}$, which originates at the point $(n, Y_0, \omega_1^{(j)})$ on the curve $\tilde{f} = 0$, whose magnitude and direction are determined by the restrictions imposed by Equation (IV-59) can be written as a sum of two perpendicular components one of which $\underline{A}_2^{(j-1)}$, which is defined in Equation (IV-60), also lies in the plane $\omega_1^{(j)} = \text{constant}$.

The dot product of the projections of $\underline{A}_2^{(j-1)}$ and $\frac{\nabla \tilde{f}_2}{|\nabla \tilde{f}_2|}$ on the n, Y_0 plane determines the normal displacement from the neutrally stable curve. The latter represents the location of finite amplitude periodic waves on an n vs. Y_0 plot.

Elimination of $\omega^{(j-1)}$ between the real and imaginary parts of Equation (IV-59) yields the following relation between $n^{(j-1)}$ and $Y_0^{(j-1)}$:

$$\left(A_r - \frac{C_r}{C_i} A_i\right) n^{(j-1)} + \left(B_r - \frac{C_r}{C_i} B_i\right) Y_0^{(j-1)} = Q_{ir}^{(j)} - \frac{C_r}{C_i} Q_{ii}^{(j)}$$

(IV-62)

The left-hand-side of the above equation represents the scalar product between the gradient of the function $g(n, Y_0) = 0$, which describes the behavior of the neutral stability curve, and the vector $\vec{A}^{(j-1)}$ which has been defined above.

To prove the above statement, Equation (IV-61b) is rewritten in the following form:

$$f_i(n, Y_0, \omega^{(j)}) = -\omega^{(j)} + h(n, Y_0, \omega^{(j)}) = 0$$

(IV-61c)

which upon substitution into Equation (IV-61a) gives:

$$f_r(n, Y_0, \omega^{(j)}) = f_r(n, Y_0, h(n, Y_0, \omega^{(j)})) = g(n, Y_0) = 0$$

(IV-63)

Differentiating $g(n, Y_0)$ with respect to $n^{(j)}$ and $Y_0^{(j)}$ gives:

$$\frac{\partial g}{\partial n^{(j)}} = \frac{\partial f_r}{\partial n^{(j)}} + \frac{\partial f_r}{\partial h} \frac{\partial h}{\partial n^{(j)}}$$

(IV-64a)

$$\frac{\partial g}{\partial Y_0^{(j)}} = \frac{\partial f_r}{\partial Y_0^{(j)}} + \frac{\partial f_r}{\partial h} \frac{\partial h}{\partial Y_0^{(j)}}$$

(IV-64b)

To define $\frac{\partial h}{\partial n^{(j)}}$ and $\frac{\partial h}{\partial Y_0^{(j)}}$ in terms of familiar quantities we take the differential of f_i and set it equal to zero:

$$\frac{\partial f_i}{\partial n^{(i)}} dn^{(i)} + \frac{\partial f_i}{\partial Y_0^{(i)}} dY_0^{(i)} + \frac{\partial f_i}{\partial \omega^{(i)}} d\omega^{(i)} = A_i dn^{(i)} + B_i dY_0^{(i)} + C_i d\omega^{(i)} = 0 \quad (\text{IV-65a})$$

The above relation can be rewritten in the following form:

$$d\omega^{(i)} = - \frac{A_i}{C_i} dn^{(i)} - \frac{B_i}{C_i} dY_0^{(i)} \quad (\text{IV-65b})$$

It immediately follows that

$$\frac{\partial h_i}{\partial n^{(i)}} = - \frac{A_i}{C_i} = \left(\frac{\partial \omega}{\partial n^{(i)}} \right)_{Y_0^{(i)}} \quad (\text{IV-66a})$$

and

$$\frac{\partial h_i}{\partial Y_0^{(i)}} = - \frac{B_i}{C_i} = \left(\frac{\partial \omega}{\partial Y_0^{(i)}} \right)_{n^{(i)}} \quad (\text{IV-66b})$$

Substitution of Equations (IV-66a,b) together with the relation

$$\frac{\partial f_r}{\partial h} = \frac{\partial f_r}{\partial \omega^{(i)}} = C_r \quad \text{into Equations (IV-64a,b) yields the following result:}$$

$$\frac{\partial g}{\partial n^{(i)}} = A_r - \frac{C_r}{C_i} A_i \quad (\text{IV-67a})$$

$$\frac{\partial g}{\partial Y_0^{(i)}} = B_r - \frac{C_r}{C_i} B_i \quad (\text{IV-67b})$$

Substitution of Equations (IV-67a) and (IV-67b) into the following relation

$$\nabla g \cdot \vec{A}_2^{(i-1)} = \frac{\partial g}{\partial n^{(i)}} n^{(i-1)} + \frac{\partial g}{\partial Y_0^{(i)}} Y_0^{(i-1)}$$

which represents the scalar product of the gradient of the neutrally stable

curve and the "two dimensional" Displacement Vector which has been defined in Equation (IV-59), yields an expression which is identical to the one given on the left-hand-side of Equation (IV-62). The derivation of this identity completes the proof of the statement which follows Equation (IV-62).

Since the magnitude of ∇g varies along the neutral stability line it will be more convenient to normalize Equation (IV-62) and to

consider the scalar product of $\underline{A}_2^{(j-1)}$ with $\frac{\nabla g}{|\nabla g|}$ which represents the unit vector which is normal to this curve. If in addition we let

$$\tilde{n}^{(j-1)} = n^{(j-1)} \epsilon^{j-1} + o(\epsilon^j)$$

$$\tilde{Y}_0^{(j-1)} = Y_0^{(j-1)} \epsilon^{j-1} + o(\epsilon^j)$$

represent the components of the two-dimensional Displacement Vector in n vs. Y_0 plot then Equation (IV-62) can be rewritten in the following form:

$$\begin{aligned} \frac{\nabla g}{|\nabla g|} \cdot \underline{A}_2^{(j-1)} &= \frac{(A_r - \frac{C_r}{C_i} A_i) \tilde{n}^{(j-1)} + (B_r - \frac{C_r}{C_i} B_i) \tilde{Y}_0^{(j-1)}}{\sqrt{(A_r - \frac{C_r}{C_i} A_i)^2 + (B_r - \frac{C_r}{C_i} B_i)^2}} \\ &= \frac{(Q_{ir}^{(j)} - \frac{C_r}{C_i} Q_{ii}^{(j)}) \epsilon^{j-1}}{\sqrt{(A_r - \frac{C_r}{C_i} A_i)^2 + (B_r - \frac{C_r}{C_i} B_i)^2}} = D^{(j)} \epsilon^{j-1} \end{aligned} \quad (IV-68)$$

where $\tilde{\underline{A}}_2^{(j-1)} = \epsilon^{j-1} \underline{A}_2^{(j-1)}$. Equation (IV-68) is accurate to $o(\epsilon^j)$.

If $n^{(j-1)}$ and $Y_0^{(j-1)}$ are not too large, the above equation implies that a small displacement from the neutral stability curve produces finite

amplitude oscillations with an amplitude ϵ for the first order solution.

Since $Q_{ir}^{(2)} = Q_{ii}^{(2)} = 0$ (this result can be easily derived from the definition of $Q_i^{(j)}$ which is given in Equation (IV-53d)), it immediately follows from Equation (IV-68) that to first order the normal displacement from the neutral stability line is identically zero. It thus follows that the vector $\underline{A}_2^{(1)}$ is either tangent to the neutral stability line or is identically zero. Accepting the second possibility we conclude that

$$n^{(1)} = y_o^{(1)} = \omega^{(1)} = 0$$

(IV-69)

When $j = 3$ $D^{(3)} \neq 0$ and it then follows from Equation (IV-68) that the normal displacement from the neutral stability curve is of $O(\epsilon^2)$. If in this case, the normal displacement, which may be positive or negative depending on the sign of $D^{(3)}$, is kept constant along the neutral line then the amplitude of the oscillations that take place in the vicinity of the $n^{(0)}$ vs. $y_o^{(0)}$ curve is inversely proportional to the square root of $D^{(3)}$. If on the other hand the amplitude of the oscillations is held constant then the normal displacement, that will produce an oscillation with such an amplitude, is directly proportional to $D^{(3)}$.

If any one of the eigenvalue perturbations is known, then the other two can be obtained by the simultaneous solution of the real and imaginary parts of Equation (IV-59). The same information could also be obtained if the angle, which is included between the vectors $\tilde{A}_2^{(2)}$ and $\nabla \tilde{f}_1$, was known.

Equation (IV-68) shows that for $j = 3$ and a small value of ϵ the normal displacement from the neutral line, which can be positive or negative depending on the sign of $D^{(3)}$, is proportional to the square of the amplitude and goes to zero as ϵ goes to zero. When this limit is approached periodic oscillations, which are characterized by infinitesimal amplitudes, can occur along the neutral line. This behavior can be best described by means of a local coordinate system in which the ordinate,

which originates at a point along the neutral line and is perpendicular to the n, Y_0 plane, represents the amplitude of the oscillations (i.e., ϵ) and the abscissa describes the normal displacement. In this coordinate system the location of periodic oscillations is described by means of a parabola which passes through the origin and whose shape depends on its location along the neutral line. The stability of these oscillations is yet to be determined.

Stability of the Periodic Waves

In the analysis of the previous sections, the solutions of the equations which describe the behavior of three-dimensional, finite-amplitude periodic waves were considered in detail. These solutions represent an equilibrium condition in which the net average energy added to the system (which in this case is a cylindrical combustion chamber), over a period of time, is identically zero. A slight disturbance of this energy balance may introduce an amplitude perturbation which can either grow or decay. If both the positive and the negative perturbations grow in absolute value the periodic solution is said to be unstable and vice versa.

To investigate the stability of these solutions an amplification factor λ , which was previously taken to be identically zero and which can be written as a power series in ϵ , must be included in the analysis. In this case, it becomes convenient to express the differential equations which describe the flow inside the combustion chamber and the nozzle in a "new" time coordinate, \bar{y} , which has the following definition:

$$\bar{y} = st = (\omega - i\lambda)t \quad (IV-70)$$

Analogous to Equation (II-9) a different time lag, \bar{Y}_0 , will now be defined:

$$\bar{Y} = s\bar{Y}_0 = (\omega - i\lambda)\bar{Y}_0 \quad (IV-71)$$

If in the analysis that preceded this section we let \bar{y} replace y , \bar{Y}_0 replace Y_0 , and $s = \omega - i\lambda$ replace ω , the final results will have the same form as in the case of periodic solutions. These final results describe the wave pattern inside the combustion chamber and the

relationship between the eigenvalue perturbations. This wave motion is no longer periodic (with time) and depending on the sign of λ these waves may grow or decay (with time). As in the analysis of periodic solutions, the eigenvalues which appear in the analysis of the aperiodic motion (i.e., n , \bar{Y}_0 and $s = \omega - i\lambda$) are all included in the boundary conditions and differential equations which control the behavior of $\tilde{F}_{(m,v,h)}$. The latter contains the first-order solution and these components of the second and third-order eigenfunction expansions which have the same time and space dependence as the first-order solution. Letting $s = \omega - i\lambda$ replace ω and repeating exactly the analysis of the previous section would yield the relationship which exists, in the case of aperiodic motion, between the eigenvalue perturbations. In this case replacing ω by s in Equation (IV-8a) shows that the existence of a nonzero amplification factor will result in the modification of the previously calculated values of n and Y_0 . In particular, if λ is of second order the modifications in n and Y_0 will also be of second order. Since finite amplitude periodic waves were found at a distance of the order of ϵ^2 from the neutral stability line, the stability analysis must be performed for eigenvalue perturbations which are of the same order (i.e., $\sigma(\epsilon^2)$). Consequently we take

$$\lambda = \lambda^{(2)} \epsilon^2$$

(IV-72)

Replacing $\omega^{(2)}$ by $s^{(2)} = \omega^{(2)} - i\lambda^{(2)}$ and $Y_0^{(2)}$ by $\bar{Y}_0^{(2)}$ in Equation (IV-52), using the definition of $\bar{Y}_0^{(2)}$ and separating the resulting equation into its real and imaginary parts yield the following result:

$$A_r n^{(2)} + \omega^{(0)} B_r \tau_0^{(2)} + C_{ir} \omega^{(2)} = Q_{ir} - \lambda^{(2)} C_{ii}$$

(IV-73a)

$$A_i n^{(2)} + \omega^{(0)} B_i \tau_o^{(2)} + C_{ir} \omega^{(2)} = Q_{ir} + \lambda^{(2)} C_{ir}$$

(IV-73b)

where

$$C_{1r} = \tau_o^{(0)} B + C$$

(IV-74)

and A , B , C and Q_1 are defined in Equations (IV-53a) through (IV-53d). Elimination of $\omega^{(2)}$ from Equations (IV-73a) and (IV-73b) and use of Equation (IV-72) yield the following relation

$$\tilde{A} \tilde{n}^{(2)} + \tilde{B} \tilde{\tau}_o^{(2)} = \tilde{Q}_1 \epsilon^2 - \lambda \tilde{C}_1$$

(IV-75)

where

$$\tilde{n}^{(2)} = n^{(2)} \epsilon^2 + o(\epsilon^3)$$

(IV-76a)

$$\tilde{\tau}_o^{(2)} = \tau_o^{(2)} \epsilon^2 + o(\epsilon^3)$$

(IV-76b)

$$\tilde{A} = C_{ir} A_r - C_{ir} A_i$$

(IV-76c)

$$\tilde{B} = \omega^{(0)} (C_{ir} B_r - C_{ir} B_i)$$

(IV-76d)

$$\tilde{Q}_i = C_{li} Q_{lr} - C_{lr} Q_{li}$$

(IV-76e)

$$\tilde{C}_i = C_{li}^2 + C_{lr}^2$$

(IV-76f)

and ϵ represents the amplitude of the oscillation which grows or decays with time. Substituting $\lambda = 0$ and $\epsilon = \epsilon^*$ (where ϵ^* is the amplitude of a periodic oscillation which neither grows or decays with time) into Equation (IV-75) gives the following relation:

$$\tilde{A}\tilde{n}^{(2)} + \tilde{B}\tau^{(2)} = \tilde{Q}_i \epsilon^{*2}$$

(IV-77)

which must be satisfied by the eigenvalue perturbations in the case of periodic oscillations.[#] The left-hand-side of Equation (IV-77) represents the normal displacement from the neutral line in an n vs. τ_0 plot. Suppose that Equations (IV-77) and (IV-75) refer to the same values of $n^{(2)}$ and $\tau_0^{(2)}$, then substitution of Equation (IV-77) into Equation (IV-75) gives:

$$\lambda \tilde{C}_i = \tilde{Q}_i (\epsilon^2 - \epsilon^{*2})$$

(IV-78)

Since \tilde{C}_i is always larger than zero, Equation (IV-78) shows that the sign of λ is determined by the sign of $\tilde{Q}_i (\epsilon^2 - \epsilon^{*2})$. Suppose $\tilde{Q}_i > 0$ then $(\epsilon - \epsilon^*) > 0$ implies that $\lambda > 0$ and $(\epsilon - \epsilon^*) < 0$ implies that $\lambda < 0$. In both of these cases $|\epsilon - \epsilon^*|$ increases with time. Suppose on the other hand that $\tilde{Q}_i < 0$, then $(\epsilon - \epsilon^*) > 0$

[#] Note that in Equation (IV-77) $\tau_0^{(2)}$ replaces $y_0^{(2)}$ which was used in the analysis of the previous section.

implies that $\lambda < 0$ and $(\epsilon - \epsilon^*) < 0$ results in $\lambda > 0$.

Consequently when $\tilde{Q}_1 > 0$, $|\epsilon - \epsilon^*|$ must always decrease with time. According to the definition presented in the beginning of this section the condition $\tilde{Q}_1 > 0$ represents the case of unstable periodic oscillations and vice versa.

It has been shown previously that $\tilde{Q}_1 \epsilon^{*2}$ equals the normal displacement from the neutral line. Thus when $\tilde{Q}_1 > 0$ unstable periodic solutions of amplitude ϵ^* can be found at a distance of $O(\epsilon^2)$ in the linearly stable region (which is outside the parabolic curve representing the neutral stability line). If on the other hand $\tilde{Q}_1 < 0$, then stable periodic solutions can be found at a distance of $O(\epsilon^2)$ in the linearly unstable region (which is inside the neutral stability curve). This situation can be best described by means of a local coordinate system which can have its origin at any point along the neutral line and whose ordinate (which is normal to the plane containing the neutral line) and abscissa represent respectively the amplitude of the oscillation and the normal displacement. In this coordinate system the locus of periodic solutions is represented by means of a parabola which passes through the origin. When $\tilde{Q}_1 > 0$ the abscissa of this coordinate system will point into the linearly stable region. In this case the amplitude of a solution which lies above or to the left of this parabola will grow in time and the amplitude of a solution which lies below or to the right of this curve will decay with time.

This situation is described in Figure 8 where "growth" and "decay" regions are distinctly separated from one another. The existence of this "growth" region brings into light the possibility of "triggering" combustion instability. When this possibility prevails, the "introduction" of any disturbance with an amplitude ϵ , which is larger than a certain threshold value, which is given by a point on the parabola which describes the locus of periodic solutions, to the system will result in unstable combustion. In this case the amplitude of the oscillations will grow with time. From physical considerations this amplitude cannot grow indefinitely. The appearance of some other nonlinear mechanisms, which become important at higher amplitude levels and which have not been discussed in the present

analysis, is expected to determine the final value reached by this amplitude. At its "new" amplitude level the oscillation may be stable or unstable and may possibly appear in a different form (e.g., rotating shock or detonation wave). The determination of the new limiting value of the amplitude and the final form attained by the wave oscillation would require a separate investigation.

In conclusion we see that when $\tilde{Q}_1 > 0$ the introduction of finite-amplitude disturbances can trigger instability while small disturbances will decay to zero.

When $\tilde{Q}_1 < 0$ the normal displacement is directed into the linearly unstable region. In this region finite-amplitude periodic waves are stable. Consequently the amplitude of any solution which lies in the linearly unstable region will tend to coincide with some point on the given parabola which indicates the locus of stable periodic solutions. Thus when $\tilde{Q}_1 < 0$ an inward displacement will produce finite-amplitude periodic waves while an outward displacement will produce a solution whose amplitude will decay to zero. (See Figure 8 for description of this case.) This case represents a situation in which finite-amplitude periodic oscillations without shock waves are possible.

In the case of small amplitude oscillation, linear analysis predicts that the amplitude of the oscillation will decay if the solution is represented by a point in the linearly stable region (which corresponds to $\tilde{Q}_1 > 0$) and it will grow if the solution lies in the linearly unstable region (which corresponds to $\tilde{Q}_1 < 0$). These conclusions are in complete agreement with the results of the nonlinear analysis presented in this section. The latter is qualitatively identical to the corresponding analysis performed by Sirignano⁴ in his investigation of longitudinal oscillations. The quantitative results which determine whether at a given point along the neutral line the displacement is in the inward or outward direction will depend, however, on the particular problem which is being considered.

CHAPTER V.

NUMERICAL EXAMPLES, DISCUSSION AND CONCLUSIONS

Introduction

The theory which had been developed in previous chapters was applied to the solution of some specific problems with the following objectives in mind:

- (a) The investigation of the effect of the Mach number of the mean flow and the chamber length upon linear transverse combustion instability.
- (b) The determination of the nonlinear wave form and its dependence upon the frequency of the oscillation, the Mach number of the steady flow and the location in the combustion chamber.
- (c) The determination of the stability of finite amplitude periodic waves.

In Chapter IV the particular case in which the combustion process takes place within an infinitesimally thin zone immediately adjacent to the injector face and the flow field inside the combustion chamber is irrotational was considered in detail. The solutions of the first three coefficients of an asymptotic series representing the wave motion inside the combustion chamber have been obtained. Considering in detail the form of each of these coefficients, it can be shown that they depend on the combustion process, the frequency of the oscillation, Mach number of the steady flow, length of the combustion chamber, the characteristics of the nozzle and so on. Because of the complicated manner in which each of these coefficients depends on any one of the above-mentioned parameters it is practically impossible, unless some additional assumptions are introduced, to obtain much insight into the problem by merely considering the analytical form of these coefficients. This problem has been resolved by obtaining numerical solutions of the wave form and the linear stability limits of several rocket engines; each of which is characterized by a different set of parameters (i.e., length, Mach number, etc.). The results of these computations will now be presented.

First-Order Results

If we assume for the moment that ϵ , the expansion parameter is very small then the amplitudes of the second and third-order solutions become negligible in comparison to the amplitude of the first-order solution. In this case the solution of the problem can be approximated by the first-order solution. This approximation is equivalent to obtaining a linearized solution of the problem. In this case the first-order solution can be used for the calculation of the linear stability limits of a rocket engine. Equation (IV-13a) represents the final outcome of such an analysis. The relations given in this equation were used to investigate the effect of the Mach number of the mean flow, the length of the combustion chamber and the frequency of oscillation upon linear combustion instability.

Because of the complicated form of the analytical solutions for $n(\omega)$ and $\tau(\omega)$, a digital computer was employed in their numerical evaluation. The computer program was divided into two parts. In the first part, the differential equations which describe the nonsteady flow field in the subsonic portion of the nozzle were integrated numerically. The nozzle throat was chosen as the initial point from where the integration proceeded in the direction of the combustion chamber. The integration was terminated when the local Mach number (in the nozzle) was equal to the Mach number of the mean flow inside the combustion chamber. The results of the numerical integration[#] were then fed into the second part of the computer program where $n(\omega)$ and $\tau(\omega)$ were calculated. In order to "run" this computer program it was necessary to specify apriori the ratio of the specific heats (γ), the frequency of the oscillation, the frequency of the corresponding acoustic mode (i.e., $S_{(\nu, h)}$), the length of the combustion chamber, the Mach number of the steady flow inside the combustion chamber and the geometry of the nozzle. Specification of the nozzle shape is equivalent to the specification of the steady-state velocity distribution in the subsonic portion of the nozzle. The latter is necessary for the numerical integration of Equation (II-192) (which describes the behavior of μ) whose coefficients depend on this velocity distribution.

[#] In our particular case (of irrotational flow) Equation (II-192), which describes the behavior of μ , was integrated numerically. As can be seen from the analysis of Chapter IV, the knowledge of μ which appears in the Nozzle Admittance Relation is necessary for the calculation of n and τ .

In this particular study, a conical nozzle with 30° half angle convergent section was used in the calculations. When the numerical calculations were originally started, algebraic expressions were used for the specification of the velocity profile in the subsonic section of the nozzle.[#] This method was found, however, to be inefficient as well as not accurate enough and was consequently changed later on in the program. Instead, it was chosen to describe the velocity profile by means of an ordinary differential equation whose integration was performed simultaneously with the integration of Equation (II-192) (for μ). A more detailed description of this method can be found in Section D of Reference 23. The nozzle shape and the ratio of the specific heats were kept unchanged in all the cases which were considered in the present investigation. Once the Mach number of the mean flow, the length of the combustion chamber and the associated acoustic frequency were specified, the computer program was "run" over varied ranges of frequencies. For a given frequency, the output of each run contained, among other quantities, the values of $n(\omega)$ and $\tau(\omega)$. These "runs" were repeated for various values of the Mach number of the mean flow, the corresponding acoustic frequency (i.e., $S_{(\nu,h)}$) and the length of the combustion chamber.

Some of the results obtained in these calculations are presented in Figures 9 through 18. These figures give theoretical predictions of the linear stability limits for rocket motors whose mean flow is characterized by Mach numbers (denoted by \bar{q} in the figures) which vary between 0.3 and 0.6 and nondimensional lengths which range from 0.5 to 3.0^{##}. In these figures SNH represents the value of the associated acoustic frequency. The actual calculations were considerably more extensive than those presented here. In addition to the results reported above, the linear stability limits of rocket engines whose mean flow is characterized by Mach

[#] See Appendix C of Reference 2 for a more detailed description of these algebraic expressions.

^{##} In the figures the length is denoted by THET which has the following definition:

$$THET = \bar{q} \cdot L_{c.c.} = \bar{q} \cdot \frac{\hat{L}_{c.c.}}{\hat{r}_{c.c.}}$$

where $\hat{}$ indicates a dimensional quantity. $L_{c.c.}$ denotes the nondimensional length of the combustion chamber.

numbers which range from 0.3 down to 0.05 have also been considered. In general, the results obtained in these calculations were qualitatively similar to the results obtained by Reardon². Consequently a detailed description of these results will not be given here. Only those points which were either not mentioned or not observed by Reardon² and other investigators in the field will now be discussed.

The following are some of the interesting results which were obtained in the calculations in the low Mach number range (i.e., from 0.3 to 0.05):

(1) In plots of n , the interaction index, vs. the frequency of the oscillation the ranges of the frequencies within which combustion instability could occur became narrower as the Mach number was reduced. The indication was that in the limit, as the Mach number of the mean flow goes to zero, instability can occur at only one particular value of the frequency. For this value of the frequency the operation of the rocket will be unstable for all values of n . The frequency at which this phenomenon occurs is equal to the corresponding acoustic frequency.

(2) Negative values of n (the interaction index) were obtained in the calculation of the linear stability limit for low values of the Mach number. Since a combustion process which is characterized by a negative value of n is physically not possible, it was concluded that in those ranges of the frequency for which n was found to be negative the operation of the engine must be stable. Scala¹² also reported the calculation of negative values of n .

A comparison of the results of the present analysis with those reported in Reference 2 shows that in the latter case the linear stability curves were characterized by higher values of $n_{\min}^{\#}$. This discrepancy will be discussed later on in this chapter. In making any comparisons it is important to remember, however, that the results reported by Reardon² and Scala¹² apply to the case of distributed combustion and rotational flow.

No restrictions regarding the magnitude of the Mach number of the mean flow or the frequency range^{##}, were imposed in the present study.

n_{\min} indicates the lowest value of the interaction index for which neutral oscillations are possible.

See footnote on page 173.

This is contrary to the investigations performed by Reardon², Scala¹² and others. Consequently, the theories developed here could be used to investigate the effects that increasing the Mach number of the steady flow and varying the frequency range may have upon the linear stability characteristics of various rocket engines. A better understanding of these phenomena can be obtained by considering in detail the results presented in Figures 9 through 18.[#]

The first interesting observation is the appearance of "loops" in plots of n vs. τ or n vs. $\omega\tau$. These "loops" appear at higher values of the frequency.^{##} For a given combustion chamber, an increase of the Mach number results in a decrease in the size of the "loops" which now appear at lower values of the interaction index. This observation suggests that these "loops" actually exist at all Mach numbers. However, in the case of flows with low Mach numbers the "loops" exist at infinity (i.e., $n = \infty$) and thus they do not appear in the linear stability plot. For a given value of $S_{(v,h)}$ the appearance of the "loops" is associated with the appearance of new instability regimes. This phenomenon can be best observed by considering the plots of n vs. the frequency. The curves presented in these plots look like modified sine curves. The first "valley" (or minimum) of this curve corresponds to the instability regime that is "associated" with the acoustic frequency of pure transverse oscillations. Subsequent "valleys" which appear at higher values of the frequency correspond to the instability regimes which are "associated" with the mixed acoustic modes. It is important to notice that as the frequency is increased these "valleys" become wider (or flatter) and they are characterized by higher values of n . The significance of this behavior will be discussed in following sections.

It has also been noted that the centers of these "valleys" are usually removed from the values of the corresponding acoustic frequencies (i.e., the pure transverse frequency or the mixed modes). Examination of Figures 9 through 14 and Figure 18 shows that increasing the length of the combustion chamber or Mach number of the mean flow results in the shifting of the various n_{\min} as well as the whole unstable region which is associated with each one of them toward lower values of the frequency.

[#] See discussion on page 187 for the meaning of the various symbols which appear in these figures.

^{##} It must be remembered that the frequency appears as a parameter in the calculation of n and τ .

Examination of Equation (IV-12a) suggests that $n_{\min.}$ occurs whenever $\omega \tau = \pi$; the latter implies that $H_i = 0$. Using this relation together with the definition of H (i.e. Equation IV-11) it can be shown that the frequency at which the interaction index is minimum indeed depends, in a very complicated way, on the shape of the nozzle, length of the combustion chamber and Mach number of the mean flow.

Similar "loops" (in n vs. τ plots) have been observed by Mitchell[#] in his investigation of the effect of increasing the Mach number upon the linear stability of the longitudinal modes. Although the shape of the "loops" appearing in the longitudinal case is somewhat different than the shape of the "loops" which can be observed in the present investigation, their qualitative dependence on the Mach number is the same.

In the foregoing discussion it has been stated that subsequent loops (i.e., those which are associated with higher values of the frequency) are characterized by higher values of the interaction index; while that portion of the curve that describes the stability limits of the pure transverse mode is represented by the lowest values of the interaction index. It can consequently be concluded that, if a rocket engine has an n -value such that it is stable with respect to the pure transverse mode, then it is unconditionally stable with respect to all the other modes (i.e., subsequent mixed modes). Similar conclusions were arrived at by Crocco and Cheng,¹ who investigated the stability of longitudinal waves. In that case, however, the mixed modes are replaced by higher longitudinal modes.

The examination of the results presented in Figures 9 through 18 yields the following conclusions regarding the effect of increasing the Mach number (hereinafter denoted by \bar{q}) on the linear stability limits.

(1) The unstable regions (in plots of n vs. τ) move from lower to higher values of τ .

(2) In plots of n vs. ω the instability regions move towards lower values of the frequency.

(3) More unstable regimes (or "loops") appear within a given range of the frequency. This is more noticeable with longer chambers.

[#] For discussion of this case see Section C in Reference 23.

(4) The value of $n_{\min.}$ slightly increases. This effect is more noticeable with combustion chambers of shorter lengths.

(5) In plots of n vs. the frequency the unstable regions become "flatter" and less distinguishable. In this plot the overall unstable region increases.

Another interesting phenomenon is the observation that increasing the length of the combustion chamber resulted in decreasing $n_{\min.}$. This can be seen best from Figure 17. Similar results were also reported in studies of the linear stability of longitudinal waves.¹ In the latter case an additional cylindrical section was inserted between the combustion chamber and the nozzle. It has also been assumed that the combustion process has been completed before the flow entered the additional section. Under these conditions the steady-state velocity distribution inside the additional section is uniform and the additional section would be regarded as either belonging to the nozzle or the combustion chamber.

A similar situation exists in our present analysis of three-dimensional waves; i.e., increasing the length of the combustion chamber does not affect the combustion process. Such an addition will generally result in decreasing the frequency of the oscillation in the combustion chamber as well as in the nozzle.[#] The Nozzle Admittance Relation depends on μ , which in turn depends on the frequency of oscillation inside the nozzle. It will be shown later that at lower frequencies μ has a greater destabilizing effect, i.e., it results in a lower value of n . It can thus be concluded that, under the described conditions, increasing the length of the combustion chamber has a destabilizing effect. It should also be mentioned that increasing the length of the combustion chamber may result in a shift of $n_{\min.}$ to a higher value of γ . Sirignano²³, who examined several different approaches for the calculation of the linear

[#] It can be shown that the nondimensional frequency inside the nozzle is directly related to the nondimensional frequency inside the combustion chamber.

stability limits of transverse oscillations, reports - that for combustion chambers which are characterized by higher Mach numbers (of the mean flow) - increasing the length of the chamber has indeed a destabilizing effect. The opposite effect is reported in the case of low-Mach-number-flows. The latter case has not been considered in the present investigation.

Disregarding the "loops", whose significance has already been discussed, the linear stability limits which were calculated in the present investigation are qualitatively similar to those presented by other investigators.[#] A closer comparison reveals, however, that the values of $n_{\min.}$ presented in the present investigation are consistently lower than the values of $n_{\min.}$ which were calculated by other investigators. For example, for Mach number of 0.3 the present analysis predicts values of $n_{\min.}$ which are in the vicinity of 0.1 while the results by Reardon² and Sirignano²³ show that $n_{\min.} \sim 0.5$.

In an effort to account for the above discrepancy it will be useful to discuss a recent publication by Cantrell and Hart²⁵. This paper investigates the requirements for neutral acoustic stability in cavities where there is a mean flow in the absence of acoustic disturbances. Through clever manipulations Cantrell and Hart have succeeded in developing a criterion for the neutral stability of an acoustic cavity which depends on first-order quantities only. When the flow inside the cavity is irrotational and no sinks or sources are present, this criterion can be expressed in the following form:

$$\left\langle \int_S dS \cdot \left\{ \hat{p}^{(n)} \hat{q}^{(n)} + \frac{\hat{p}^{(n)2}}{\bar{\rho} \bar{c}^2} + \hat{q}^{(n)} \hat{p} \left(\frac{\hat{q}^{(n)}}{\bar{\rho}} \cdot \hat{q}^{(n)} \right) + \frac{\hat{p}^{(n)} \hat{q}^{(n)} \left(\hat{q}^{(n)} \cdot \hat{q}^{(n)} \right)}{\bar{c}^2} \right\} \right\rangle = 0 \quad (V-1)$$

[#] See, for example, the results presented in References 2 and 23.

where $\langle \rangle$ indicates a time average of a given quantity and $\hat{\cdot}$ denotes a dimensional quantity. Above integration is performed over the surface of the acoustic cavity. Equation (V-1) represents the balance that must exist between the various forms of energy which are crossing the boundaries of the cavity when the oscillations inside the cavity are neutrally stable. Nondimensionalizing it with respect to steady-state quantities and using the following definition of the admittance relation

$$y_b = \frac{q^{(1)}}{p^{(1)}} \quad (V-2)$$

Equation (V-1) can be rewritten in the following form:

$$\left\langle \int_S \vec{dS} \cdot \left\{ p^{(1)2} \vec{y}_b + \frac{1}{\gamma} \vec{q} p^{(1)2} + \gamma p^{(1)2} \vec{y}_b (\vec{q} \cdot \vec{y}_b) + p^{(1)2} \vec{q} (\vec{q} \cdot \vec{y}_b) \right\} \right\rangle = 0$$

(V-3)

In order to use this formula for estimating the value of n_{\min} , it will be first necessary to obtain the appropriate expressions for $\vec{y}_b \cdot \vec{dS}$. Using the definitions of $\xi^{(1)}$, $\kappa^{(1)}$ and $\pi^{(1)}$ together with Equation (III-30) results in the following expression for the dot product of the admittance relation at the injector end with the unit vector which

is normal to this surface. This unit vector points in the outward direction.

$$\vec{y}_{I.E.} \cdot \vec{n} = \frac{\bar{q}}{\bar{p} \delta} \left(1 - n^{(0)} \delta \bar{p} (1 - e^{-i\omega \tau}) \right)$$

(V-4)[#]

Similarly, using Equation (II-189) it can be shown that the dot product of the nozzle admittance with a unit vector axially directed into the nozzle can be written in the following form:

$$\vec{y}_n \cdot \vec{n} = - \frac{\bar{q}}{\bar{p} \delta} \frac{\mu \hat{K}}{\bar{q}_n^2 \mu + i\omega}$$

(V-5)

where \hat{K} is a proportionality constant that must be introduced to account for the fact that different nondimensionalizing schemes are used in the combustion chamber and in the nozzle.

Since the cylindrical walls of the combustion chamber are perfectly rigid, their admittance is identically zero. Assuming that ω , δ and \bar{p} are $O(1)$, it immediately follows from Equations (V-4) and (V-5) that $\vec{y}_{I.E.} \cdot \vec{n}$ and $\vec{y}_n \cdot \vec{n}$ must be $O(\bar{q})$. Using this fact and neglecting terms of $O(\bar{q}^2)$ (which is justified when \bar{q} is reasonably small) Equation (V-3)

A careful review of the analysis presented in Chapter III will show that $\vec{S}^{(1)}$, which appears in Equation (III-30), actually equals $\frac{\vec{u}^{(1)}}{\bar{q}} \cdot \vec{n}$ where \vec{n} is a unit vector normal to the injector face pointing into the combustion chamber. In the present analysis we are interested in the dot product of $\frac{\vec{u}^{(1)}}{\bar{q}}$ with \vec{n} , which equals $-\frac{\vec{u}^{(1)}}{\bar{q}} \cdot \vec{n}$. The latter is used in the derivation of Equation (V-8).

can be rewritten in the following approximate form:

$$\int_{\text{INJECTOR}}^{\text{END}} ds \langle P_r^{(1)} \rangle \left(R_e (\vec{n} \cdot \vec{y}_{I.E}) - \frac{1}{8} \bar{q} \right) + \int_{\text{NOZZLE}}^{\text{ENTRANCE}} ds \langle P_r^{(1)2}(z=z_e) \rangle \left(R_e (\vec{n} \cdot \vec{y}_n) + \frac{1}{8} \bar{q} \right) = 0$$

(V-6)

In developing the above relation dS was replaced by $\vec{n} dS$ where \vec{n} is a unit vector normal to the surface of the combustion chamber; \vec{n} is directed in the outward direction.

Since the admittance relations at the two ends of the combustion chamber and the steady-state velocity distribution are of $\sigma(\bar{q})$ it can be expected that the solution representing the first-order pressure distribution inside the combustion chamber can be represented in the following form

$$P_{c.c.} = P_{acoustic} + \sigma(\bar{q}) \quad (V-7)$$

It immediately follows from Equations (V-6) and (V-7) that replacing $P_{c.c.}$ by $P_{acoustic}$ in Equation (V-6) introduces errors which are of $\sigma(\bar{q}^2)$. The latter are considered negligible compared to terms of $\sigma(\bar{q})$ in the present analysis. Substituting the following well-known solution of the acoustic pressure

$$P_{acoustic} = \text{Re} \left\{ A \cos \left(\frac{\pi z}{L} \right) J_n \left(S_{(v,h)} r \right) \cos(\nu t) e^{i\omega t} \right\} \quad (V-8)$$

into Equation (V-6), and assuming that the area of the injector face equals the area at the nozzle entrance in addition to remembering that in the

present analysis \bar{q} stays constant throughout the combustion chamber, the following result is obtained:

$$\int dS \frac{1}{2} A^2 J_n^2 \cos^2 n\theta \left\{ \operatorname{Re} \left\{ \vec{n} \cdot \vec{y}_{I,E} \right\} + \operatorname{Re} \left\{ \vec{n} \cdot \vec{y}_w \right\} \right\} = 0$$

INJECTOR
END

(V-9)

or equivalently

$$\operatorname{Re} \left\{ \vec{n} \cdot \vec{y}_{I,E} \right\} + \operatorname{Re} \left\{ \vec{n} \cdot \vec{y}_w \right\} = 0$$

(V-10)

It is interesting to note that the condition for neutral stability of small-amplitude waves inside the particular combustion chamber under consideration involves only the real parts of the admittance relations. Substitution of Equations (V-4) and (V-5) into Equation (V-10), use of the following relations

$$\bar{p} = 1 \quad ; \quad \hat{k} = 1 + \sigma(\bar{q}^2) \quad ; \quad \omega \tau = \pi$$

and neglecting terms of $\sigma(\bar{q}^2)$ yields the following result:

$$n_{\min} = \frac{1}{2\gamma} \left(1 - \frac{\mu_i}{\omega} \right)$$

(V-11)

Since γ as well as ω (the latter being the frequency of oscillation in the nozzle) are always positive, Equation (V-11) suggests that when μ_i is positive, increasing μ_i or decreasing ω will result in lowering the value of n_{\min} . From stability considerations it is consequently desired to have $\mu_i < 0$. Since μ_i (which is obtained from the numerical integration of Equation (II-192) is indirectly dependent upon the Mach number of the mean flow inside the combustion chamber[#], it immediately

[#] This is true because the numerical integration of Equation (II-192), which controls the behavior of μ , is terminated when the local Mach number in the nozzle equals the Mach number of the mean flow inside the combustion chamber.

follows that $n_{\min.}$ is implicitly dependent upon \bar{q} .

To obtain a better understanding of the manner in which $n_{\min.}$ depends on the various parameters which appear in this problem the following table has been constructed from the available numerical data.

TABLE I #

ω_N	\bar{q}	μ_i	$n_{\min. \text{ exact}}$	$n_{\min. \text{ approx.}}$
1.30316	0.3	0.5747	0.1607	0.2333
1.51106	0.5	0.2797	0.2942	0.3395
1.57769	0.6	0.1857	0.3232	0.3675
1.30867	0.5	0.5171	0.5136	—
1.30802	0.6	0.4832	0.348	—

The numbers tabulated in the above table apply to the case in which $L = \frac{\bar{L}}{r_{c.c.}} = \frac{1}{2}$. $n_{\min. \text{ exact}}$ and $n_{\min. \text{ approx.}}$ are respectively the values of

$n_{\min.}$ which were obtained by use of a computer and Equation (V-11). Rows 1, 2 and 3 describe the case where $\omega\tau = \pi$ while rows 4 and 5 were included for comparison purposes. It should also be mentioned that the exact computations indicate that $n_{\min.}$ does not always occur at $\omega\tau = \pi$ and may occur somewhere else in the vicinity of this number.

The results listed in rows 1, 2 and 3 show that in all these cases $\mu_i > 0$. We can thus conclude that in the particular cases which have been investigated the nozzle acts as a destabilizing device. Physically this implies that the gases in the nozzle are "pumping" work into the combustion chamber. Since work equals the product of the pressure and axial component of the velocity, the phasing which exists between these quantities at the nozzle entrance will determine whether work is being done on or by the gases inside the combustion chamber. This phasing is determined by the flow conditions inside the nozzle and by the manner in which the waves entering the nozzle are partially transmitted through the nozzle and partially

The data presented in rows 4 and 5 does not describe cases when $\omega\tau = \pi$. Consequently the values listed under $n_{\min. \text{ exact}}$ do not represent minimum values of the interaction index.

reflected at the nozzle walls.

Further examination of the results tabulated in rows 1, 2 and 3 of Table I shows that μ_i decreases and ω_N increases as \bar{q} increases. Consequently $\frac{\mu_i}{\omega_N}$ decreases as \bar{q} increases and according to Equation (V-11) $n_{\min. \text{ approx.}}$ should increase. This result is in agreement with the exact numerical solution given by $n_{\min. \text{ exact}}$. We thus see that Equation (V-11) is at least qualitatively correct.

Equation (V-11) shows that if μ_i is very small or possibly equal to zero then $n_{\min.} \sim \frac{1}{2\gamma} = \frac{1}{2.4} = 0.416$. This value of $n_{\min.}$ is closer to the values of $n_{\min.}$ reported by Reardon² and Sirignano²³. These investigations considered, however, the case of distributed combustion[#] and consequently they cannot be treated by the approximate analysis which was presented here.

In view of what has been said above it can be concluded that it is the particular theoretical model which had been considered in this study that caused the reduction in the calculated values of $n_{\min.}$ (or n). It should also be mentioned that Crocco⁶, after considering the more general case of distributed combustion, has arrived at similar conclusions; that is, in the case of transverse oscillations the nozzle has a destabilizing effect. In the more general case, the destabilizing effect of the nozzle is, however, small.

On the basis of the discussions presented in this section it can be concluded that

(a) As the Mach number of the mean flow and the length of the combustion chamber are increased, the mixed modes (which are associated with the geometry of the combustion chamber) tend to become unstable.

(b) n -values which give stability with respect to the pure transverse mode unconditionally guarantee the stability of the engine with respect to the mixed modes.

[#] The flow field describing the distributed combustion case is generally rotational and it contains mass and energy sources as well as momentum sinks (due to droplet drag).

(c) Increasing \bar{q} in the range $0.3 \leq \bar{q} \leq 0.6$ while keeping $L_{c.c.}$ constant is stabilizing.

(d) Increasing L while keeping \bar{q} constant is destabilizing.

(e) Boundary conditions in general and the Nozzle Admittance Relation in particular are important in the determination of the stability limit of rocket engines.

(f) Under given conditions certain nozzles can act as destabilizing devices. Consequently in analyses of this nature the effect of the nozzle cannot be neglected.[#]

Before leaving this section it should be repeated that the examples considered in the above analysis represent most unstable conditions; thus the solutions of these examples yield conservative estimates of the linear stability limits of real rocket engines.

Nonlinear Wave Form

The results of the first and second-order analyses, which were performed in Chapter IV, were combined to calculate the nonlinear pressure wave form. In the present investigation the standing mode of oscillation has been considered.

The following expression

$$P(\phi, \psi, \theta, \gamma) = \bar{p} \delta (\tilde{\pi}'' \epsilon + \tilde{\pi}''' \epsilon^2)$$

was used in the numerical calculations. In this expression

$$\tilde{\pi}''' = \tilde{p}'''(\phi) \cos \theta J_1(s_{u,v} \sqrt{\frac{\gamma}{\gamma_u}}) e^{i\gamma}$$

and the solutions for $\tilde{p}^{(1)}(\phi)$ and $\tilde{\pi}^{(2)}$ are respectively given by Equations (IV-15b), (IV-45), (IV-46a) and (IV-46b). Inspection of

[#] Due to lack of knowledge of the correct form of the Nozzle Admittance Relation Cantrell and Hart and other investigators in the field have completely neglected the effect of the nozzle in their analysis of rocket instability problems.

Equation (IV-15b) shows that the solution of $\tilde{p}^{(1)}(\phi)$ is given by a single expression which depends on steady-state and first-order quantities only. The solution of $\tilde{p}^{(1)}(\phi)$ could be obtained, without much difficulty, as a by-product of the same computer program which was utilized in the calculations of the linear stability limits.

As can be seen from Equations (IV-45), the solution of $\tilde{\pi}^{(2)}$ is available as an eigen-function expansion. The evaluation of each of the coefficients appearing in this expansion (i.e., $\tilde{p}_{(km,n\nu,q)}^{(2)}(\phi)$) requires the knowledge of the first as well as the second-order solutions and must be done with the aid of a digital computer.

Because of the complex form of the analytical solutions of the second-order coefficients, the numerical evaluation of each one of them had to be performed in several steps. In the first step $A_{(n\nu,q)}$, $B_{(n\nu,q)}$ and $C_{(n\nu,q)}$ were computed; these quantities appear in Equations (IV-46a) and (IV-46b) and are defined in Appendix C. Next, in order to obtain the nonlinear Nozzle Admittance Relation, the differential equations which describe the first and second-order flows inside the nozzle had to be simultaneously integrated (numerically). Once the above information had been obtained, it was "fed" into a separate computer program where the solutions of the second-order coefficients were finally obtained. Because of the nature of the computations which were performed in this program, the evaluation of each of the second-order coefficients required considerable amounts of computer time.[#]

Theoretically, obtaining the solution of $\tilde{\pi}^{(2)}$ would require the knowledge of the coefficients $\tilde{p}_{(km,n\nu,q)}^{(2)}(\phi)$ for all the allowable combinations of the subscripts $(km,n\nu,q)$. In practice, however, if the convergence of the series under consideration is sufficiently rapid, the calculation of only the first few coefficients may yield a reasonably accurate solution. In our case, analysis of the behavior of the second-order coefficients (i.e., $\tilde{p}_{(km,n\nu,q)}^{(2)}(\phi)$) had shown that the calculation of

[#] In practice, obtaining the correct form of the second-order Admittance Relations, which were necessary for obtaining the solutions of

$\tilde{p}_{(2,0,1)}^{(2)}(\phi)$, $\tilde{p}_{(0,0,1)}^{(2)}(\phi)$, $\tilde{p}_{(1,2,1)}^{(2)}(\phi)$ and $\tilde{p}_{(0,1,1)}^{(2)}(\phi)$, required the simultaneous numerical integration of 34 ordinary differential equations. Most of the computer time which was necessary for the complete evaluation of the four coefficients mentioned above was consumed during this numerical integration.

only those coefficients which are characterized by a subscript q which varied between 0 and 5 resulted in a good approximation for $\tilde{\Pi}^{(2)}$. The contribution of all the other coefficients could be neglected.

Some of the results obtained in this investigation are presented in Figures 19 through 32 which give plots of pressure vs. time.[#] Two combustion chambers having identical lengths but different steady-state velocities (i.e., $\bar{q} = 0.3$ and $\bar{q} = 0.5$) have been considered. The effect of various parameters, such as the location inside the combustion chamber, the amplitude of the primary wave (i.e., ϵ) and the frequency of the oscillation have been studied.

In Figures 20 through 26 pressure variations for the case $\bar{q} = 0.3$ are presented. These figures give:

- (a) A comparison of the linear and the nonlinear pressure wave forms (see Figure 20).
- (b) Comparisons of the nonlinear pressure wave forms at the nozzle entrance and the injector face (see Figures 21 through 23 and note that each of these figures is characterized by a different frequency of oscillation).
- (c) Cross plots which describe: 1) the effect that the changing of the frequency of oscillation, at a given location, has upon the pressure oscillation (see Figures 24 and 25) and 2) the behavior of the pressure at a given frequency and various tranverse locations (see Figure 26).

Figures 27 through 32 describe the case for which $\bar{q} = 0.5$ and in addition contain plots which are similar to those presented in Figures 20 through 26.

Figure 19 shows experimental recordings of pressure distributions for the standing first tangential mode. These plots, which were obtained from Reference 2, were recorded during an actual firing of a sector motor. A comparison of the curve, denoted by P_1 , with the theoretical nonlinear wave form of Figure 20 (or any of the other figures) shows an excellent qualitative agreement. The curve denoted by P_2 (which was recorded at a pressure node) should be compared with the corresponding curve (i.e.,

[#] In these figures the pressure is plotted against $y = \omega t$. The latter has been used as a time coordinate throughout the analysis. Also note that these figures give the pressure at the combustion chamber wall where $r_{c.c.} = 1$.

for which $\theta = \frac{\pi}{2}$) of Figure 26. The small discrepancies between these two curves could be attributed to the fact that the theoretical plots do not include any third-order or higher-order effects. Also turbulence has an effect.

Analysis of the nonlinear wave forms shows that their behavior depends on the values of ϵ , ω , \bar{q} as well as location at which they are being considered.

One interesting observation is that increasing the Mach number of the steady flow while keeping all the other parameters constant resulted in an increase in the amplitude of the oscillation. Because of this fact the wave forms for $\bar{q} = 0.3$ were plotted with $\epsilon = 0.3$ while the wave forms for $\bar{q} = 0.5$ were plotted with $\epsilon = 0.1$. A comparison of Figures 28 and 30 or Figure 30 and any one of Figures 21 through 23 clarifies this point.

Sirignano,⁴ who performed similar calculations for the longitudinal mode of oscillation, reports that increasing the value of ϵ resulted in wave forms which became multivalued in the region of maximum amplitude. He suggests that, in the case of longitudinal oscillations, increasing the value of ϵ beyond a certain limiting value will result in shock formation. In the case of three-dimensional oscillations, increasing ϵ resulted in larger amplitudes of the nonlinear wave forms but no multivaluedness has been observed. To obtain physically meaningful results (i.e., positive pressure) it was necessary, however, to limit the allowable values of ϵ .

This behavior is in accordance with the conclusions of Maslen and Moore³. These authors have shown that while for the case of longitudinal oscillations without combustion a periodic solution without shock waves is not possible it is nevertheless possible to have finite-amplitude continuous periodic transverse waves.

Analysis of those figures which compare the pressure distribution at the injector face and the nozzle entrance[#] shows that the nonlinear wave forms are similar at these two locations. This suggests that qualitatively the shape of the nonlinear wave form must be the same throughout the combustion chamber. A study of the PHI plots also shows that in

[#] These figures are denoted as PHI plots.

all cases the pressure at the nozzle entrance slightly leads the pressure at the injector face. This behavior is probably caused by the phasing that is introduced through the use of a finite length nozzle.

Analysis of the PHI and OMEGA[#] plots indicates that the amplitudes of the oscillations are strongly dependent on the frequency. These plots suggest that for values of ω , which are smaller than $S_{(v,h)}$, the amplitude of the oscillation is larger at the injector face. It seems, however, that at a certain value of ω , which is close to $S_{(v,h)}$, a reversal of this situation takes place and for values of ω which are larger than $S_{(v,h)}$, the amplitudes of the oscillations become larger at the nozzle entrance.

In all the cases considered in the calculations, the correction to the mean pressure distribution was positive. Different behavior was observed in the analysis of longitudinal oscillations⁴ where the sign of the correction to the mean pressure distribution was found to depend on the frequency.

An analytical criterion for the determination of the stability of finite-amplitude, three-dimensional periodic waves has been derived in Chapter IV. Because of the complicated form of this criterion and the limitation of present day computers^{##} no specific examples have been considered.

Concluding Remarks

The non-steady flow conditions inside combustion chambers of liquid-propellant rocket engines have been considered. The existence of three-dimensional continuous waves which are periodic in time and have amplitudes of finite size has been proven.

These solutions were expressed as power series (in terms of ϵ) and the analytical solutions of the coefficients up to (ϵ^3)

In the Omega plots, pressure oscillations which are characterized by different frequencies are compared to one another.

The determination of the stability of finite amplitude waves requires the complete knowledge of the first and second-order solutions. Because of the large number of variables that are involved in such a computation, a computer program that was specifically written for this purpose could not be "fitted" into a 7094 IBM computer.

were derived. The latter are strongly dependent on the form of the boundary-conditions that are imposed at the two ends of the combustion chamber. In particular, the importance of knowing the correct form of the Nozzle Admittance Relation has been emphasized. The effect of the combustion process, which was described with the aid of Crocco's time-lag postulate, was introduced through the injector-end boundary condition.

Inspection shows that the calculated solutions (up to $\sigma(\epsilon^3)$) are periodic with respect to time as well as transverse location. Analysis of the differential equations which describe the wave motion inside the combustion chamber suggests that this periodic behavior is exhibited by all orders of the solution.

The frequency of oscillation was assumed to depend on the amplitude of the primary solution (i.e., ϵ) and the correction to the frequency was found to be of $\sigma(\epsilon^2)$. This result is in agreement with other theoretical treatments of similar problems.[#] It has also been found experimentally that the numerical value of the frequency of oscillation of finite amplitude waves is close to the numerical value of the acoustic frequency.

In its given form the derived solution can be considered as consisting of a basic pure mode, which is represented by the first-order solution, and higher-order solutions which cause a distortion of the basic mode. If ϵ is small, the first-order solution provides adequate representation of the wave-motion and is also capable of predicting the onset of combustion instability.

The first-order solution was used to study the manner in which increasing the Mach number of the mean flow affected the linear stability limits. The results of this analysis showed that increasing \bar{q} resulted in shifting the unstable region, which is associated with the pure transverse acoustic mode, to higher values of the time-lag. In addition new unstable regions appeared. The latter are related to instabilities of the mixed modes. Increasing \bar{q} also resulted in a very small increase of the value of n_{\min} .

The stability of finite amplitude waves was analyzed, and the possibility of "triggering" combustion instability has been proven

[#] See, for example, References 3 and 4.

theoretically. In this connection it should be mentioned that "continuity" exists between the predictions of the linear and nonlinear theories; that is, as $\epsilon \rightarrow 0$ the results of the nonlinear analysis are in complete agreement with the predictions of linear theories. In obtaining the above results a "continuity" between the linear and nonlinear mechanisms of energy addition has been assumed. Such an approach cannot, however, describe the behavior of the waves when the mechanism of energy addition is strictly nonlinear (e.g., energy addition that is associated with droplet shattering).

The possibility of "triggering" instability has been demonstrated experimentally in the case of transverse oscillations. Typical results are shown in Figure 33.[#] The latter was obtained by studying the manner in which the "introduction" of "pulses" of varying magnitudes affected the stability of a rocket engine.

Suggestions for Further Work in the Field.

Additional understanding of the problems associated with nonlinear transverse instability could be attained by attempting to solve the following problems:

(1) The study of the stability of finite-amplitude periodic oscillations when the flow inside the combustion chamber and the nozzle is rotational and all the combustion process is assumed to be concentrated within an infinitesimally thin zone.

(2) The study of the more realistic case in which the combustion process is assumed to be distributed throughout the combustion chamber.

(3) Study the effects of motion in the transverse plane on nonlinear instability as related to changes in the mixing process.

It has been mentioned previously that the possibility of "triggering" combustion instability has been verified in the case of transverse oscillations. It would be interesting, however, to obtain a close verification of the theory. This could possibly be done by obtaining the values of n , τ (i.e., the location along the neutral stability line) of

[#] Note the similarity between Figures 33 and 2 and see discussion on page 7 .

a given engine[#] and then studying the effect of pulsing on this particular engine. The results obtained in such an experiment could then be compared with the theoretical prediction, regarding the stability of finite-amplitude waves, that were obtained with these particular values of n and τ . Such an experiment could also serve as a basis for establishing the applicability of the time-lag concept to problems connected with nonlinear instabilities.

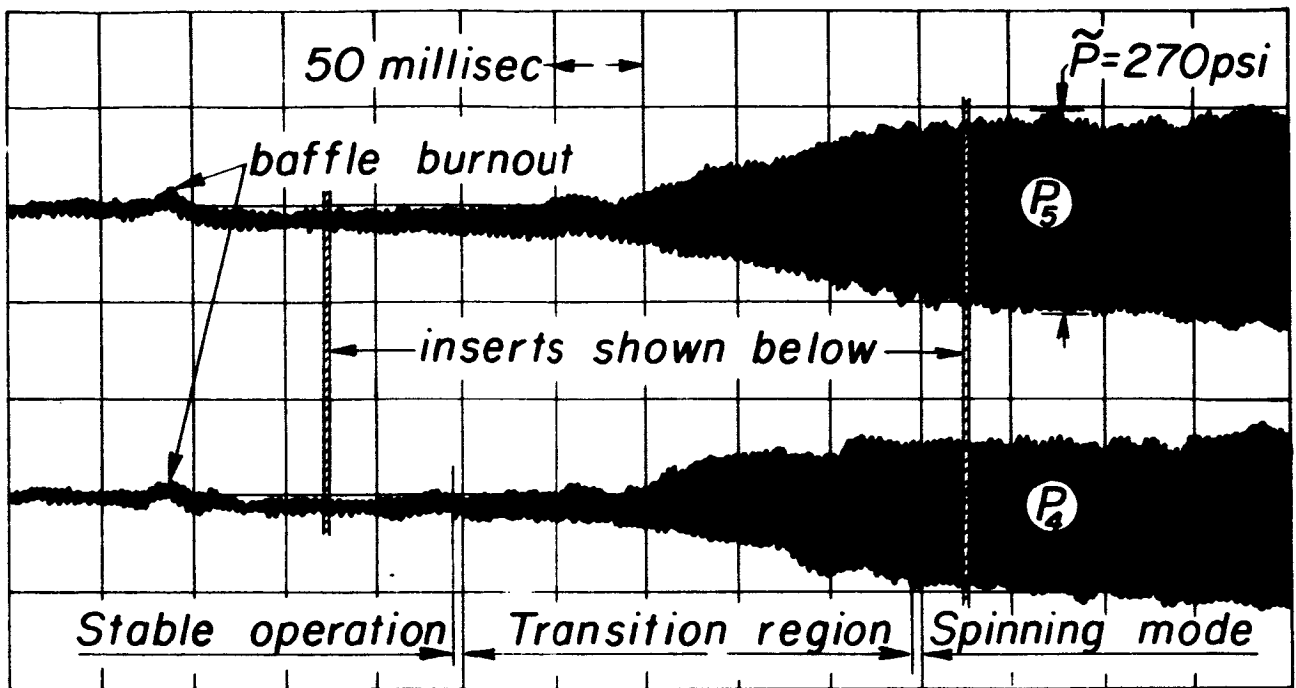
It is hoped that the results presented in this thesis together with the solutions of the suggested problems will lead towards a better understanding of the phenomena associated with nonlinear combustion instability. Such an understanding is a prerequisite for further development of the science of rocketry.

[#] Reardon² used a variable angle sector motor for the experimental determination of n and τ .

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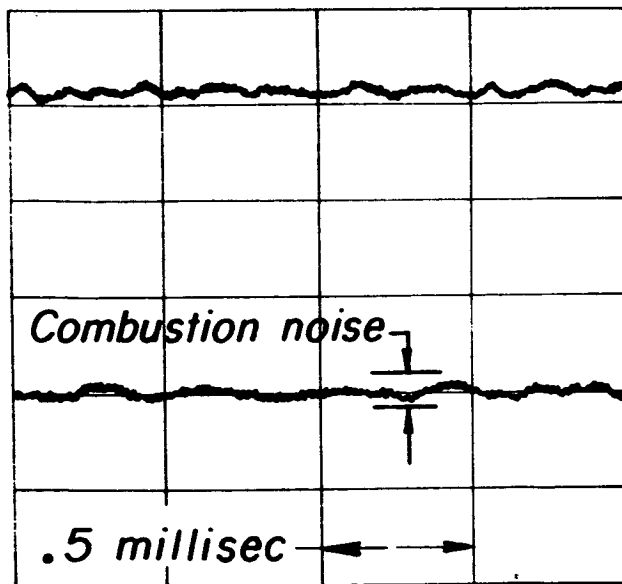
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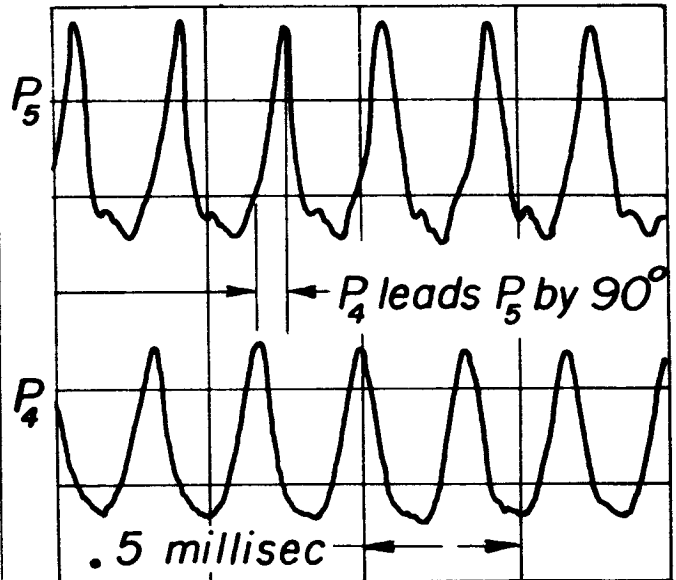


Stability record after baffle burnout

time →



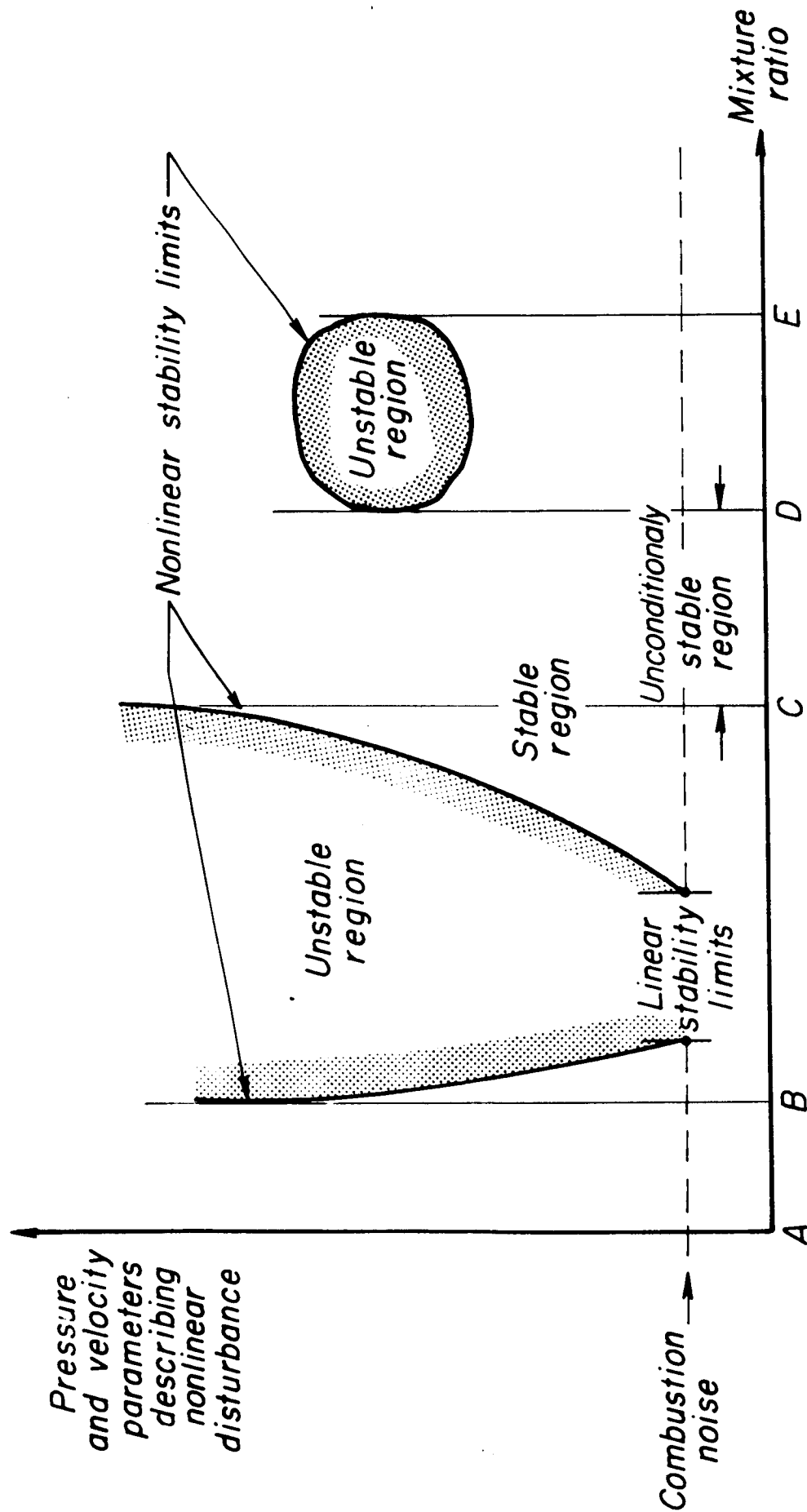
Stable operation



Spinning first tangential mode

P_4 and P_5 are pressure recording taken on the same circumference (located 1" from the injector face) while being displaced by 90° from one another. In this case the engine was linearly unstable.

Figure 1



Schematic diagram of nonlinear stability limits

Figure 2

Coordinate system used for the solution of the oscillatory nozzle flow

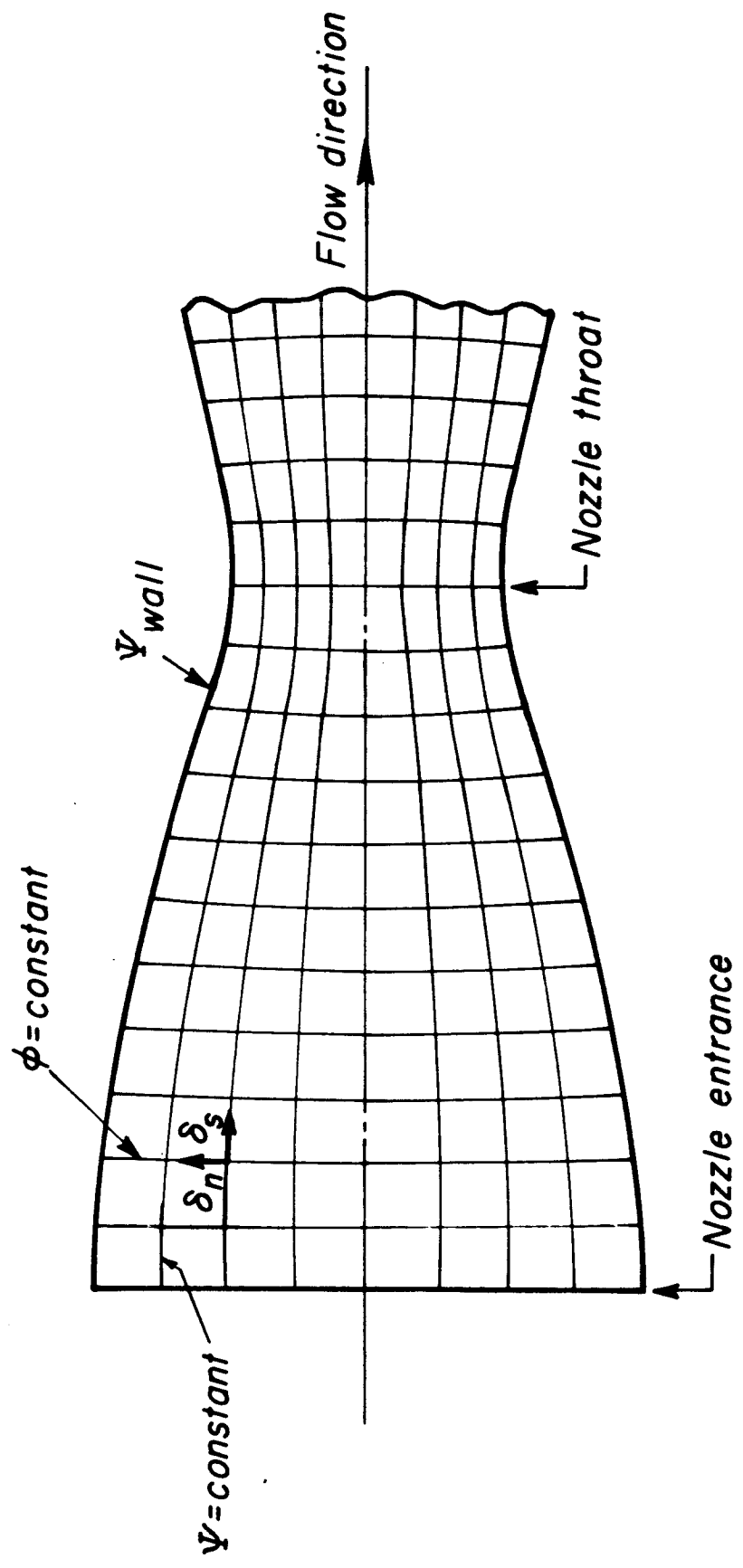


Figure 3

*Mass balance on differential volume element in
cylindrical coordinates*

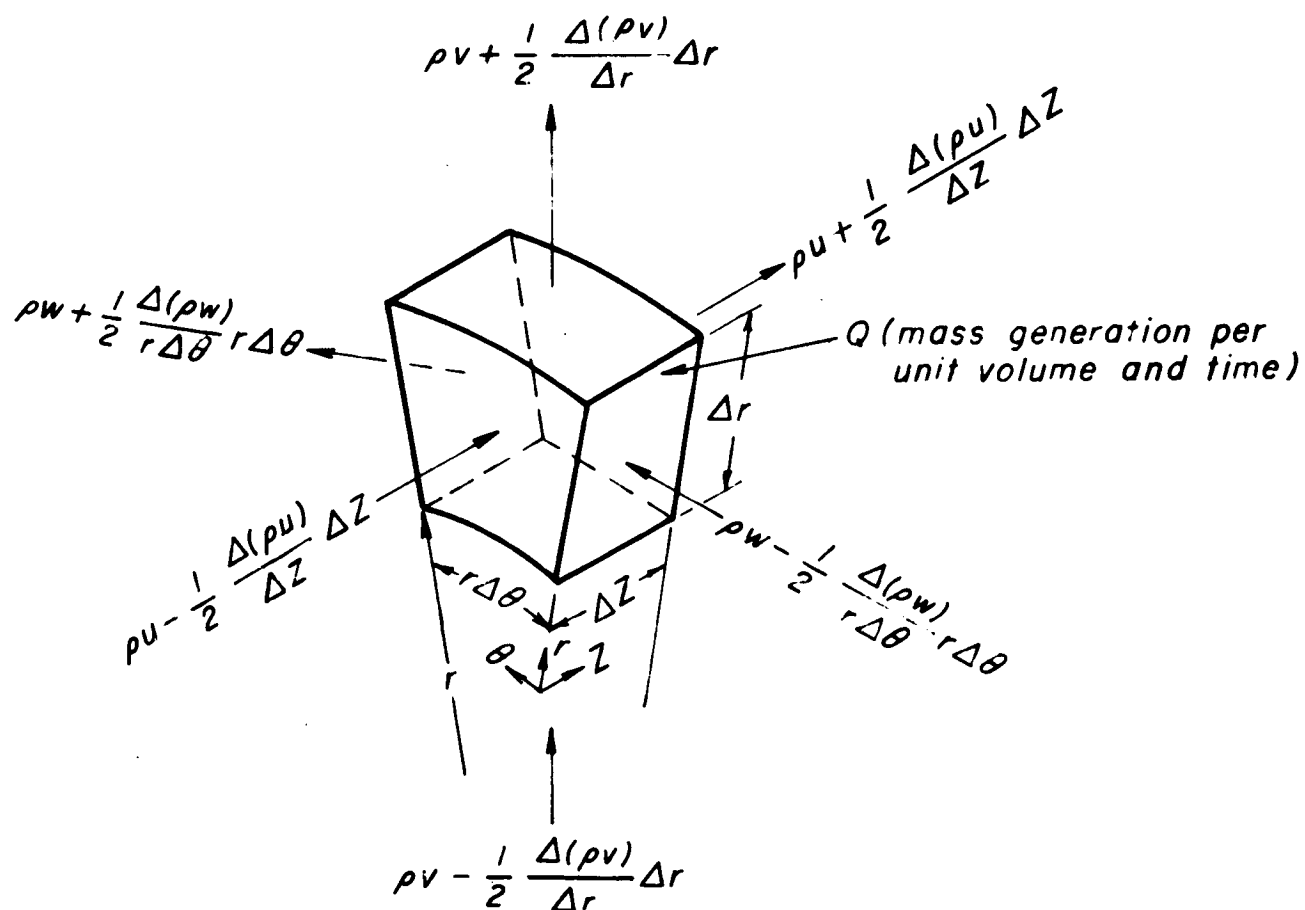


Figure 4

Limiting process transforming arbitrary volume element
into a plane at $Z = \text{constant}$

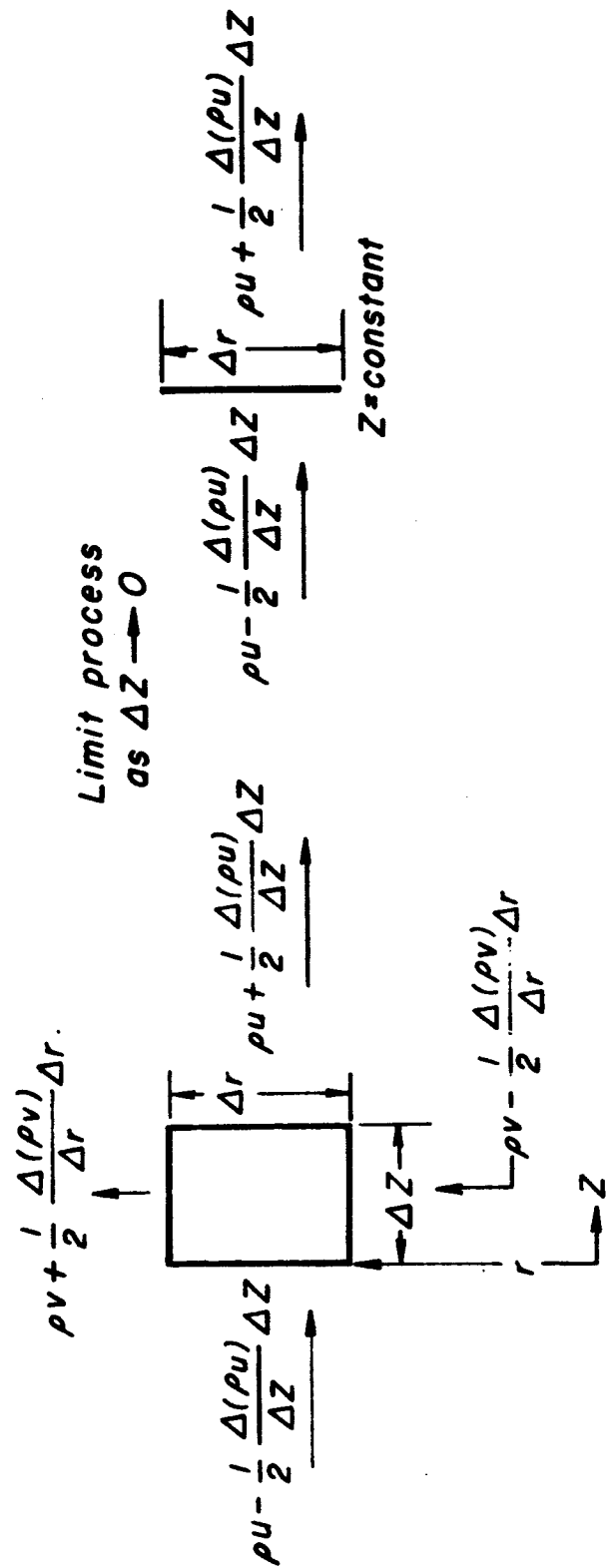


Figure 5

Schematic diagram of engine under consideration

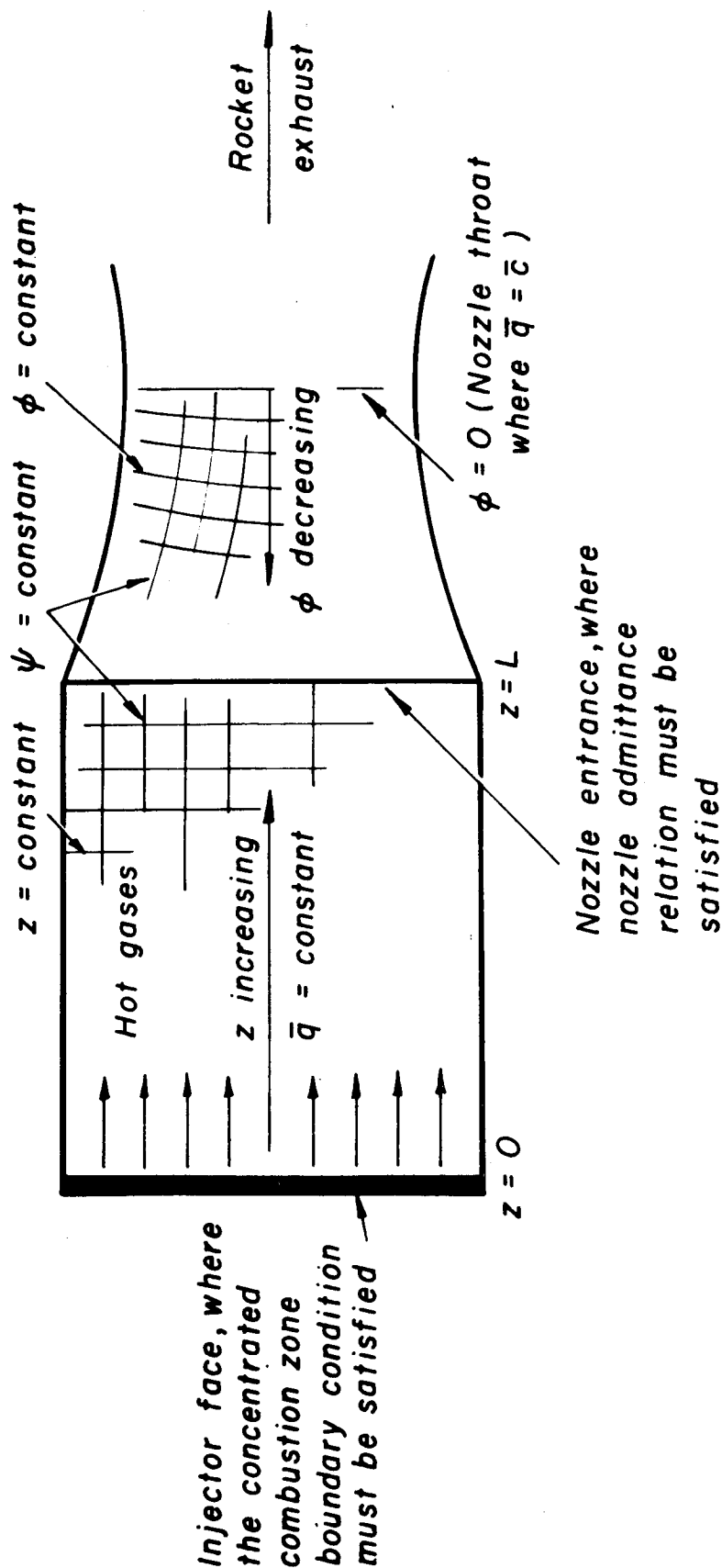


Figure 6

Geometric description of the concept of normal displacement

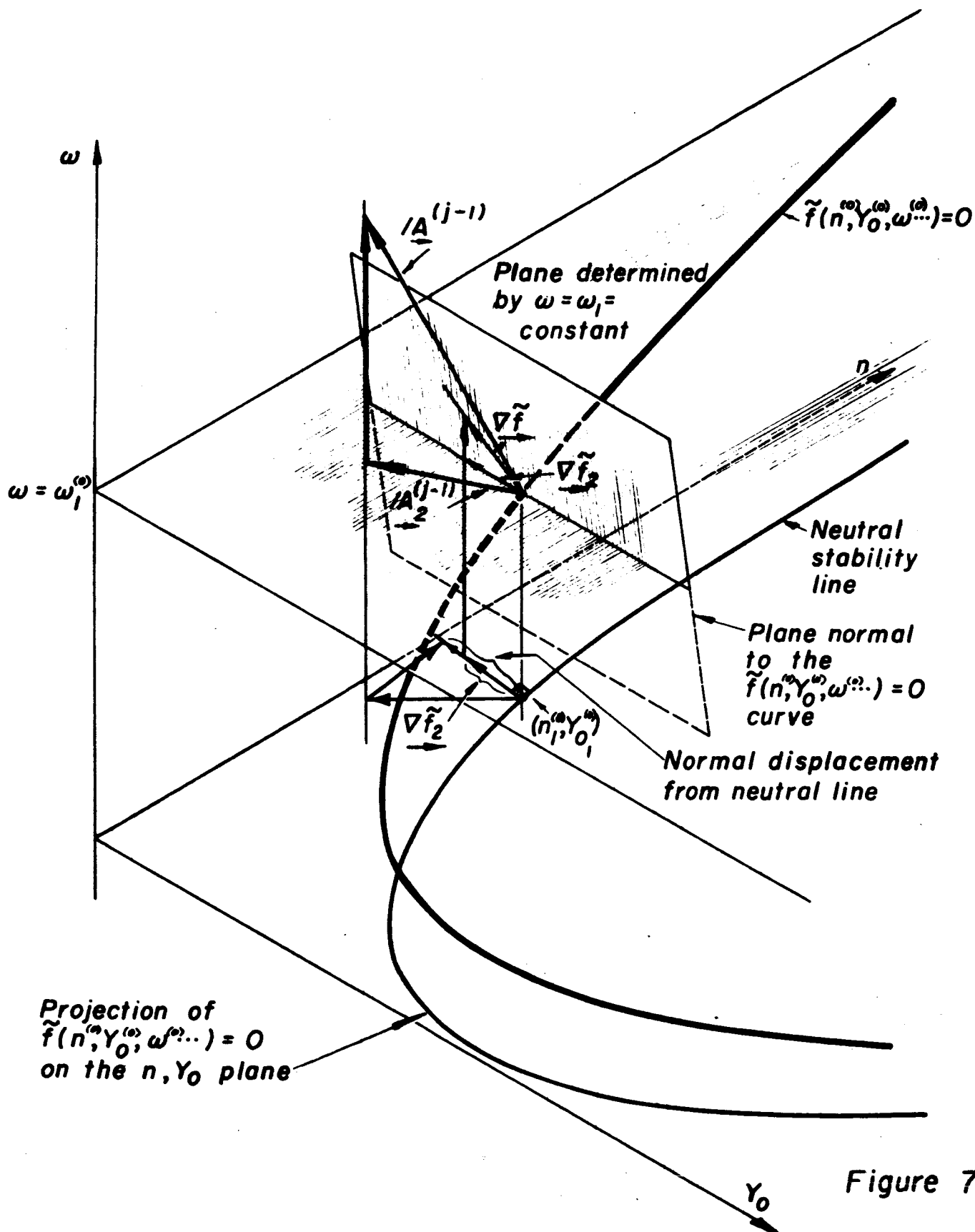


Figure 7

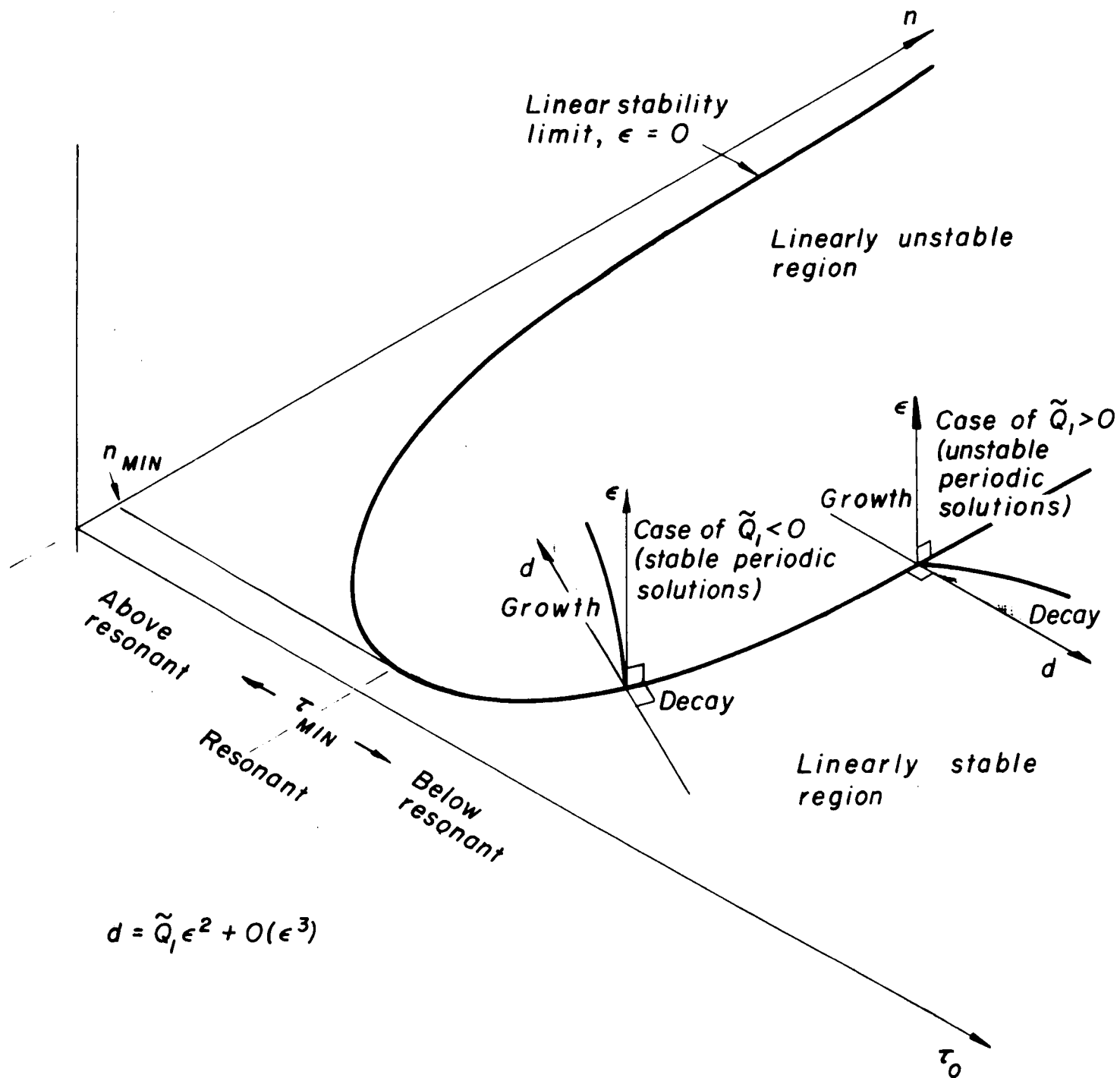


Figure 8

Neutral stability limits for the case:
 $\bar{q} = 0.3$, $THET = 0.150$, $SNH = 1.84129$

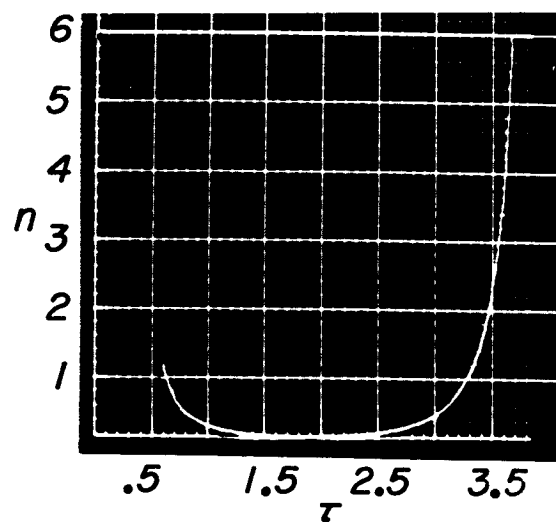
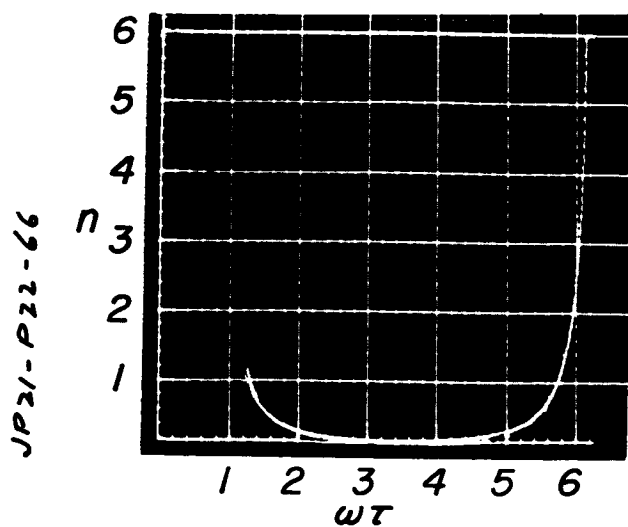
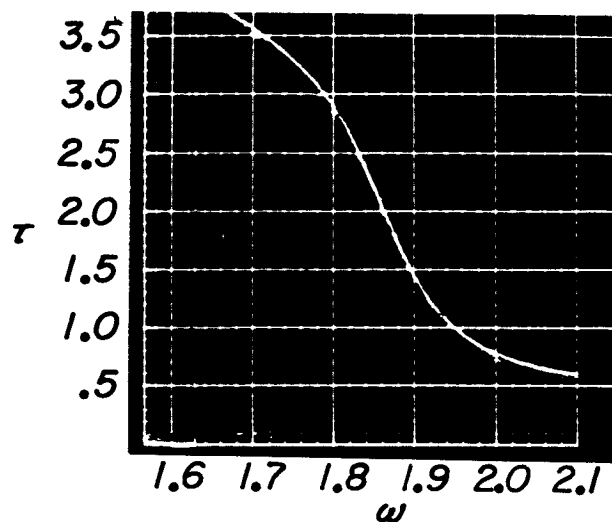
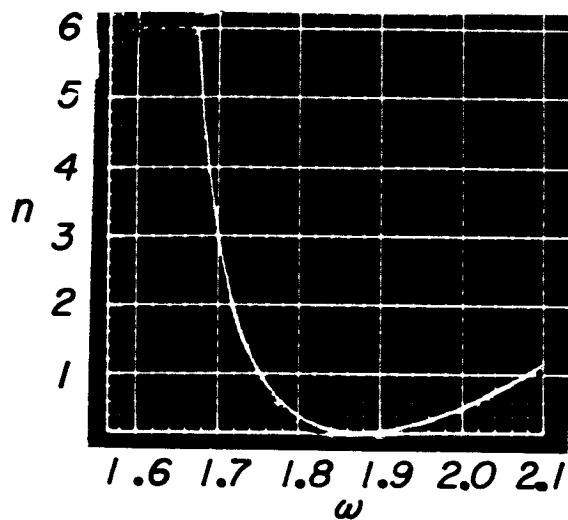


Figure 9

Neutral stability limits for the case:
 $\bar{q} = 0.3$, $THET = 0.9$, $SNH = 1.84129$

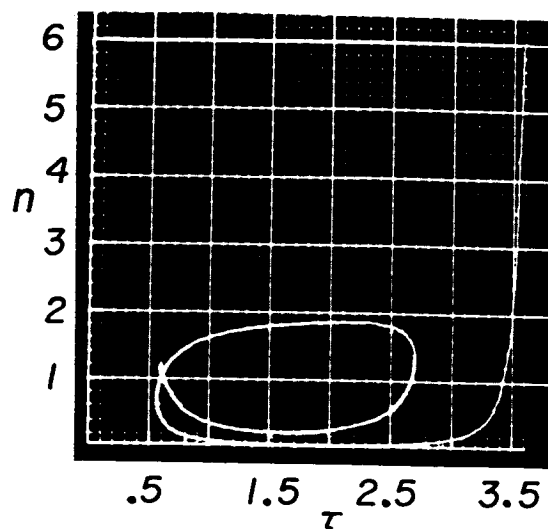
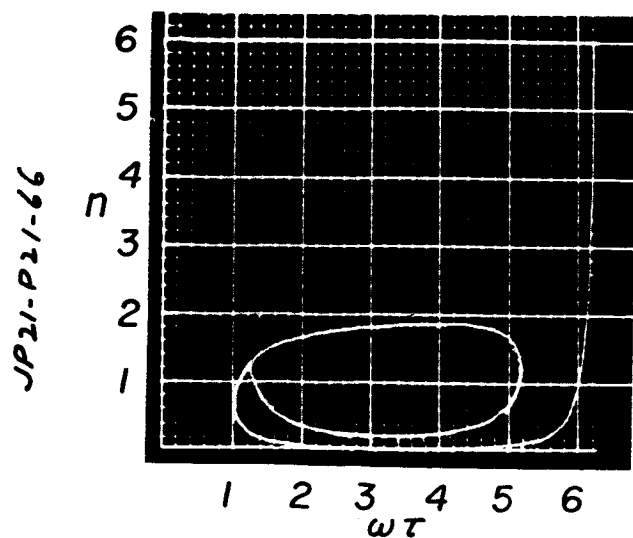
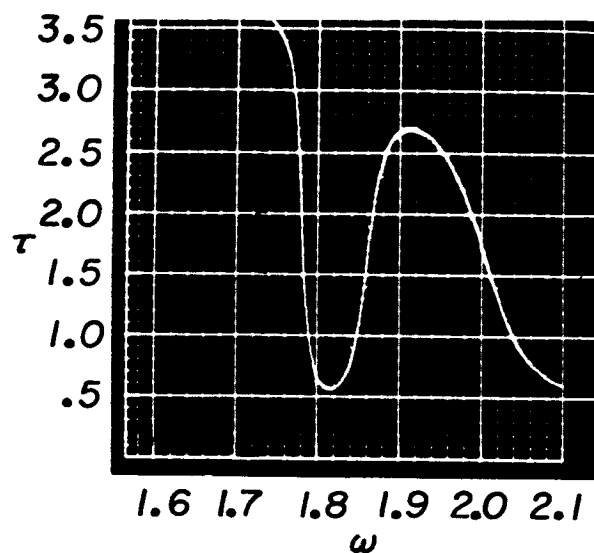
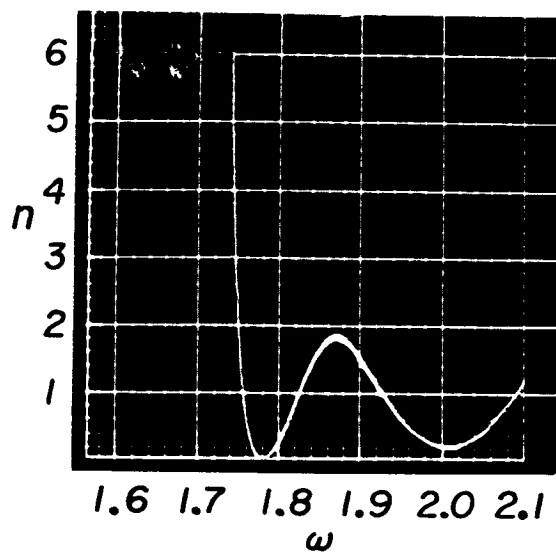
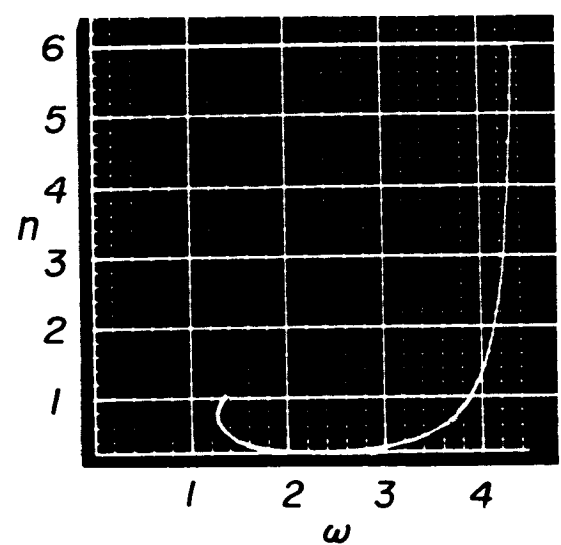
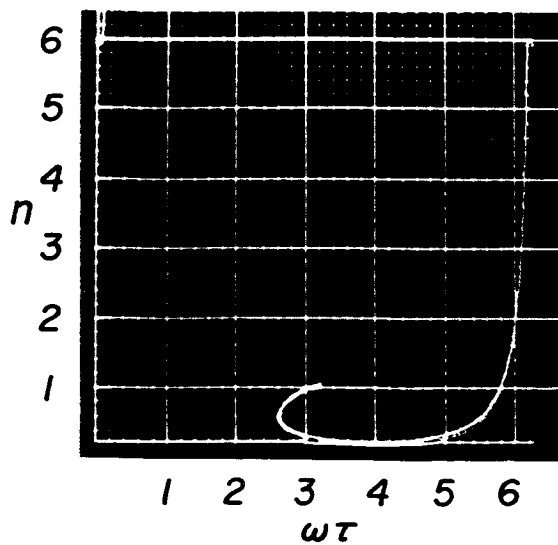
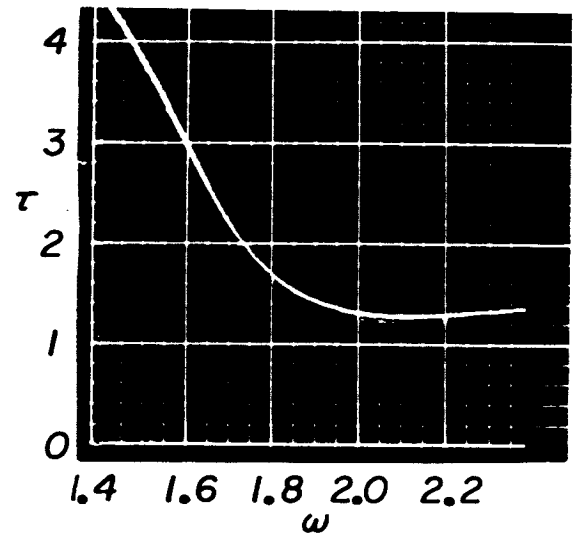
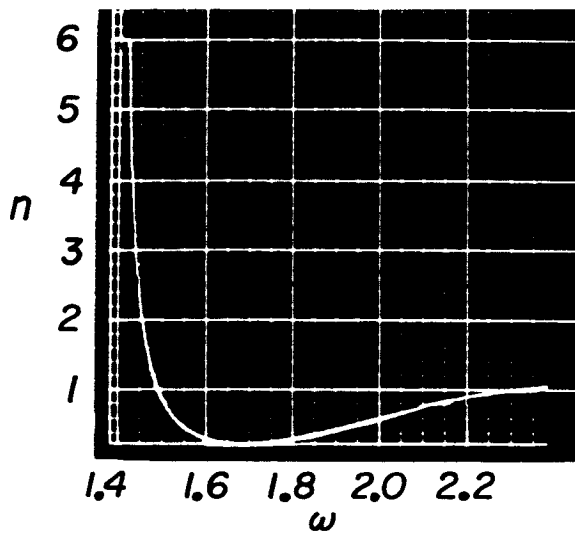


Figure 10

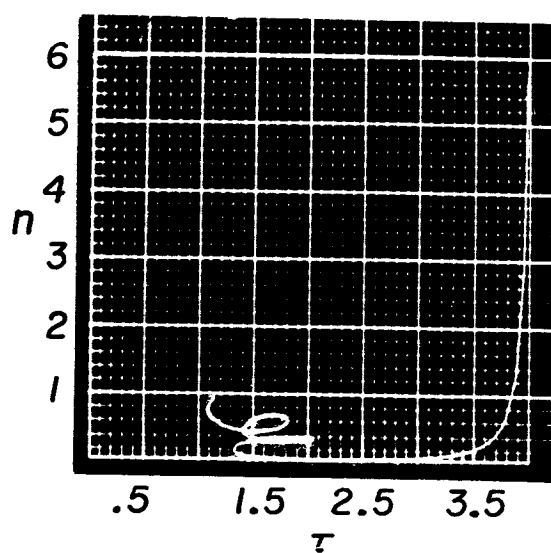
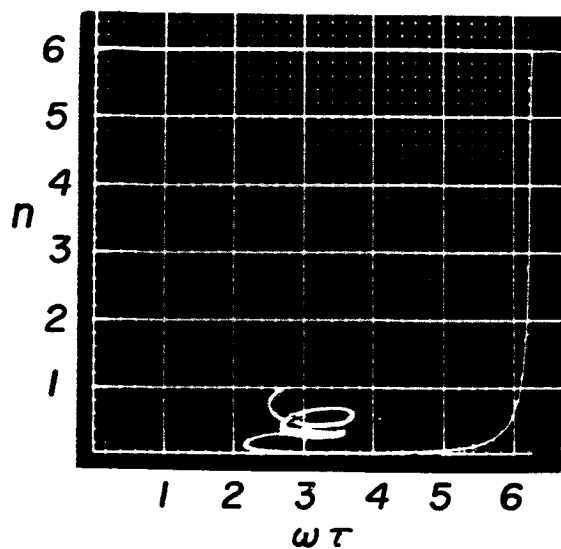
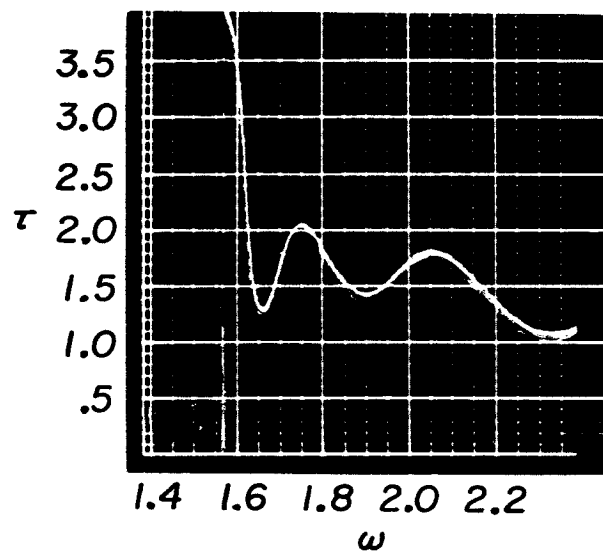
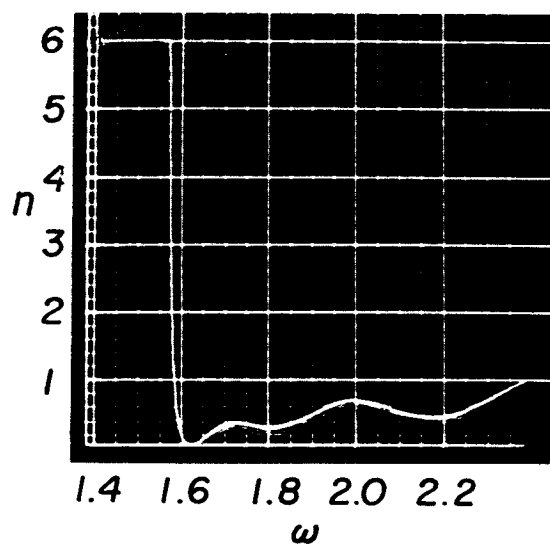
Neutral stability limits for the case:
 $\bar{q} = 0.5$, $THET = 0.25$, $SNH = 1.84129$



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Figure 11

Neutral stability limits for the case :
 $\bar{q} = 0.5$, $THET = 1.5$, $SNH = 1.84129$



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Figure 12

Neutral stability limits for the case:
 $\bar{q} = 0.6$, $THET = 0.3$, $SNH = 1.84129$

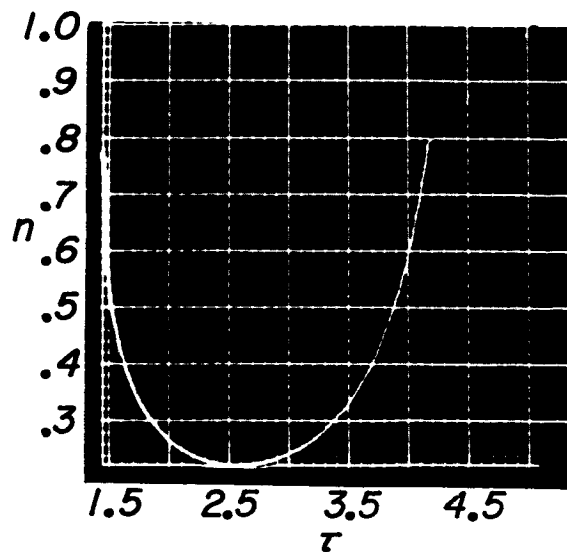
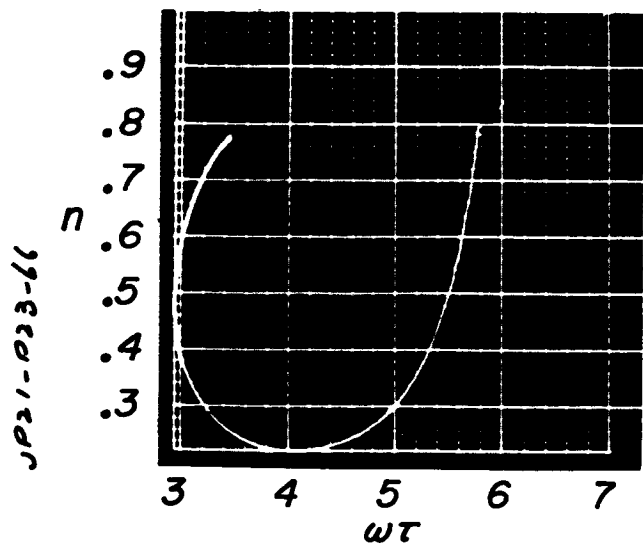
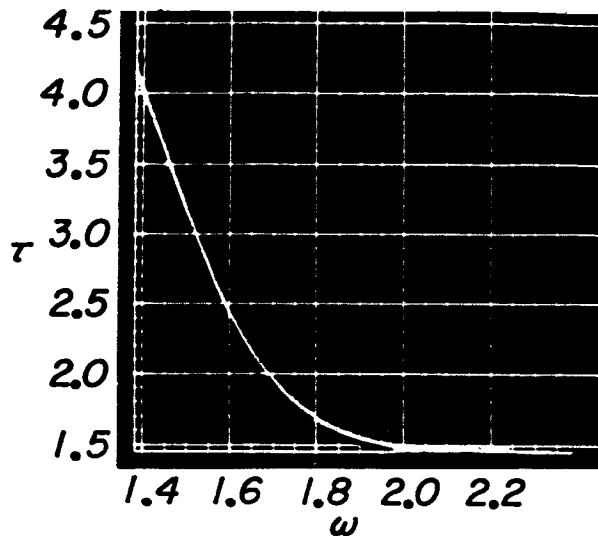
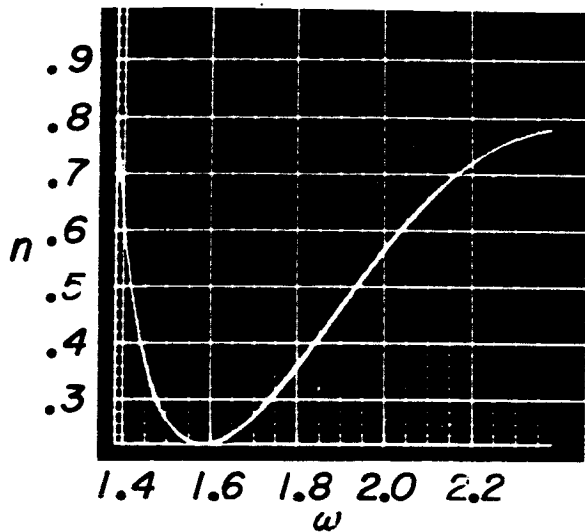


Figure 13

Neutral stability limits for the case:
 $\bar{q} = 0.6$, $THET = 1.8$, $SNH = 1.84129$

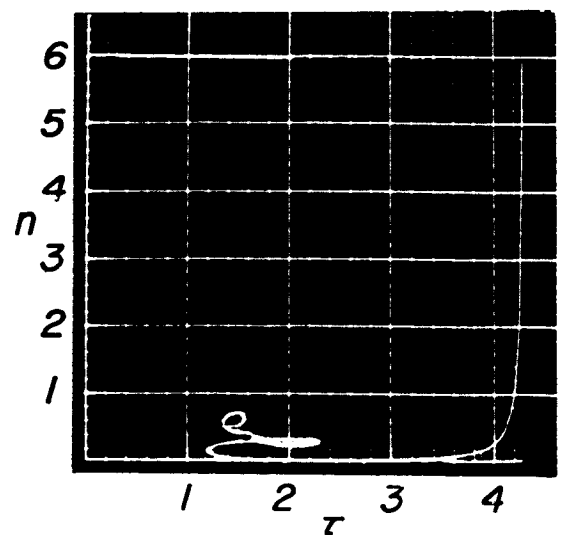
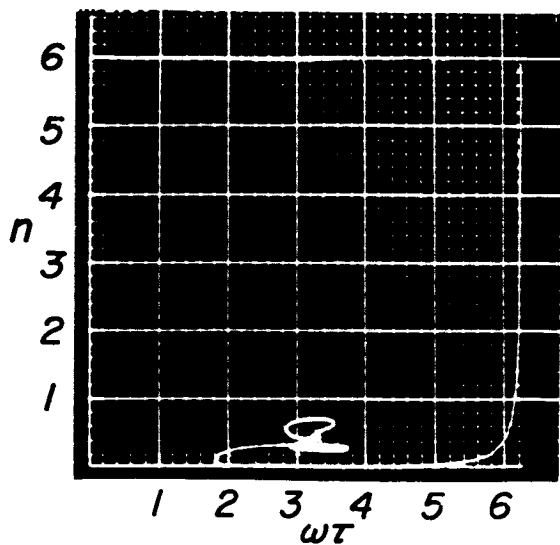
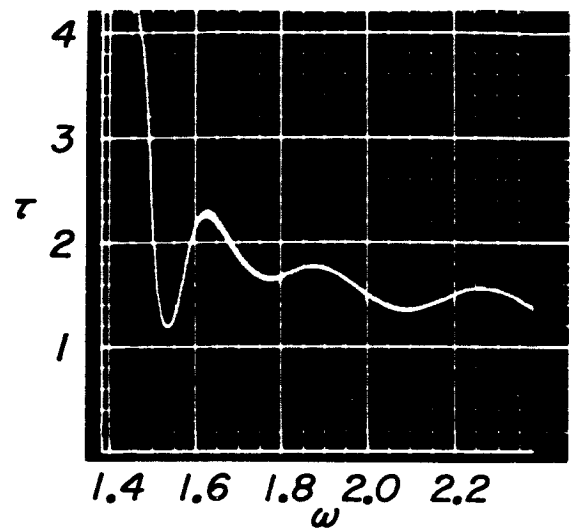
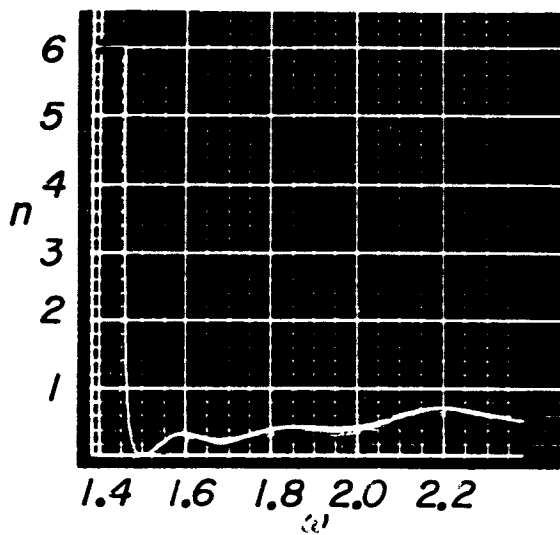
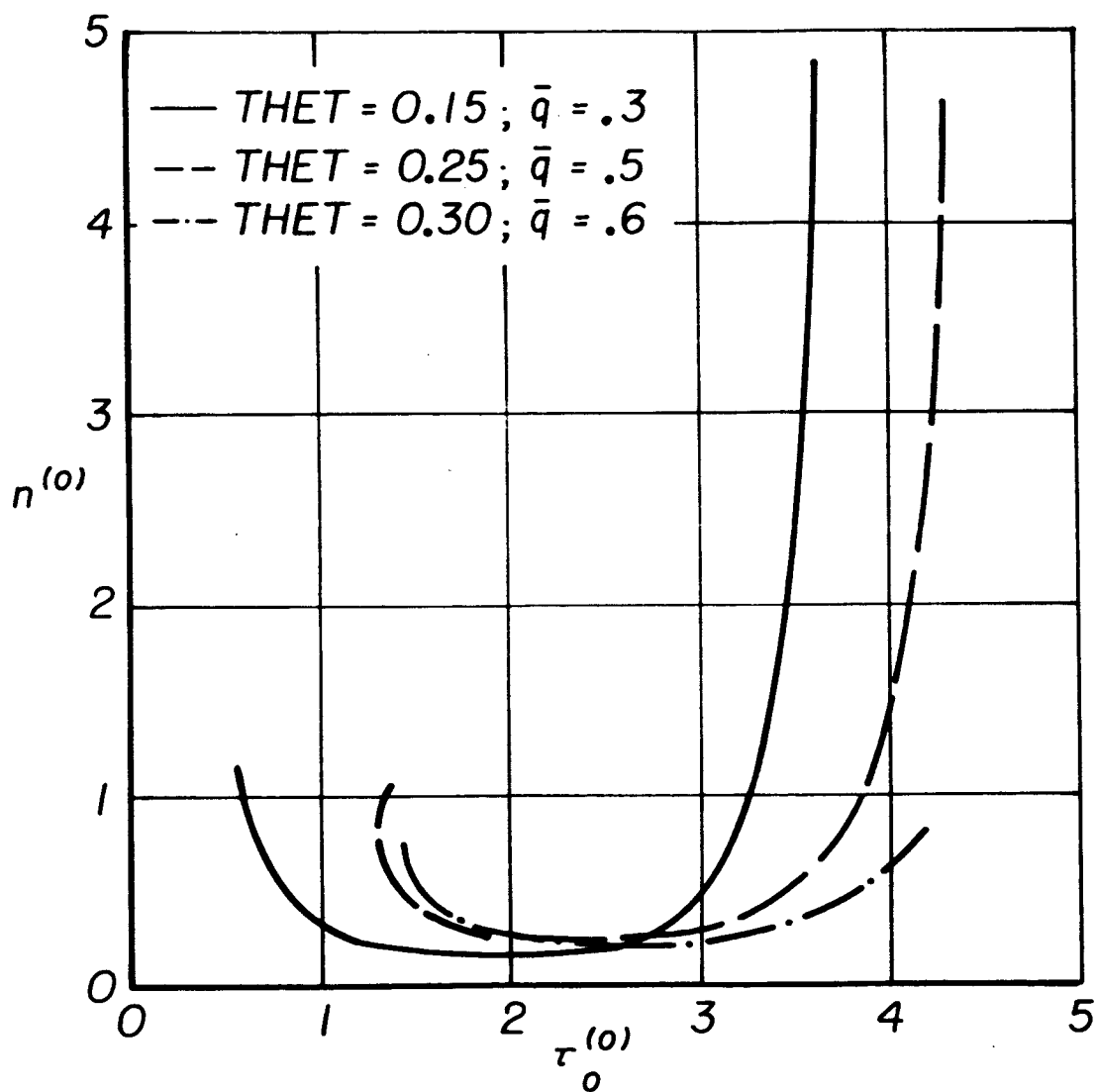


Figure 14

Comparison of the linear stability limits of combustion chambers which have identical lengths and different steady - state velocity



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Figure 15

Comparison of the linear stability limits of combustion chambers which have identical lengths and different steady-state velocity

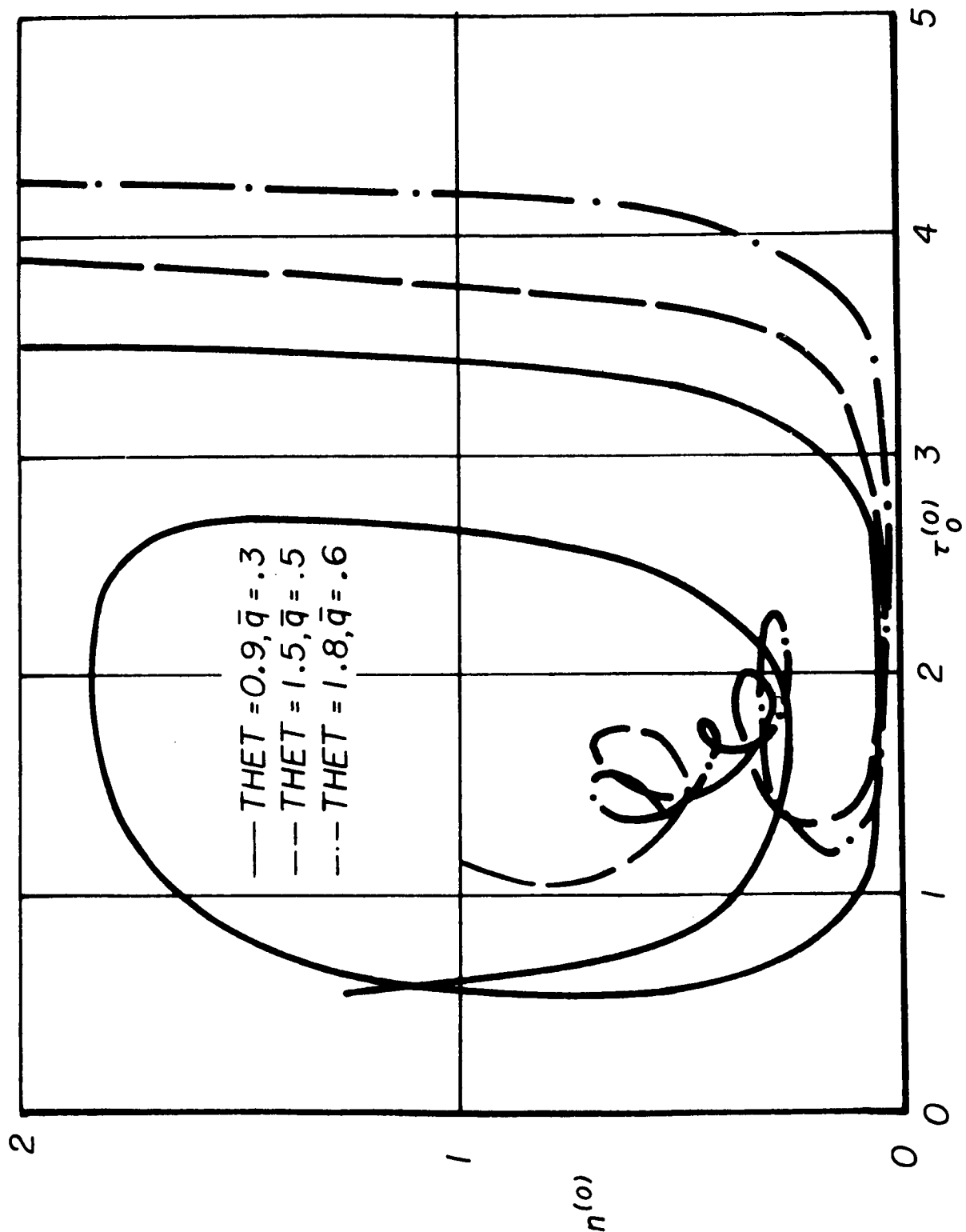


Figure 16

Comparison of the linear stability limits of combustion chambers which have the same steady - state velocity and different lengths

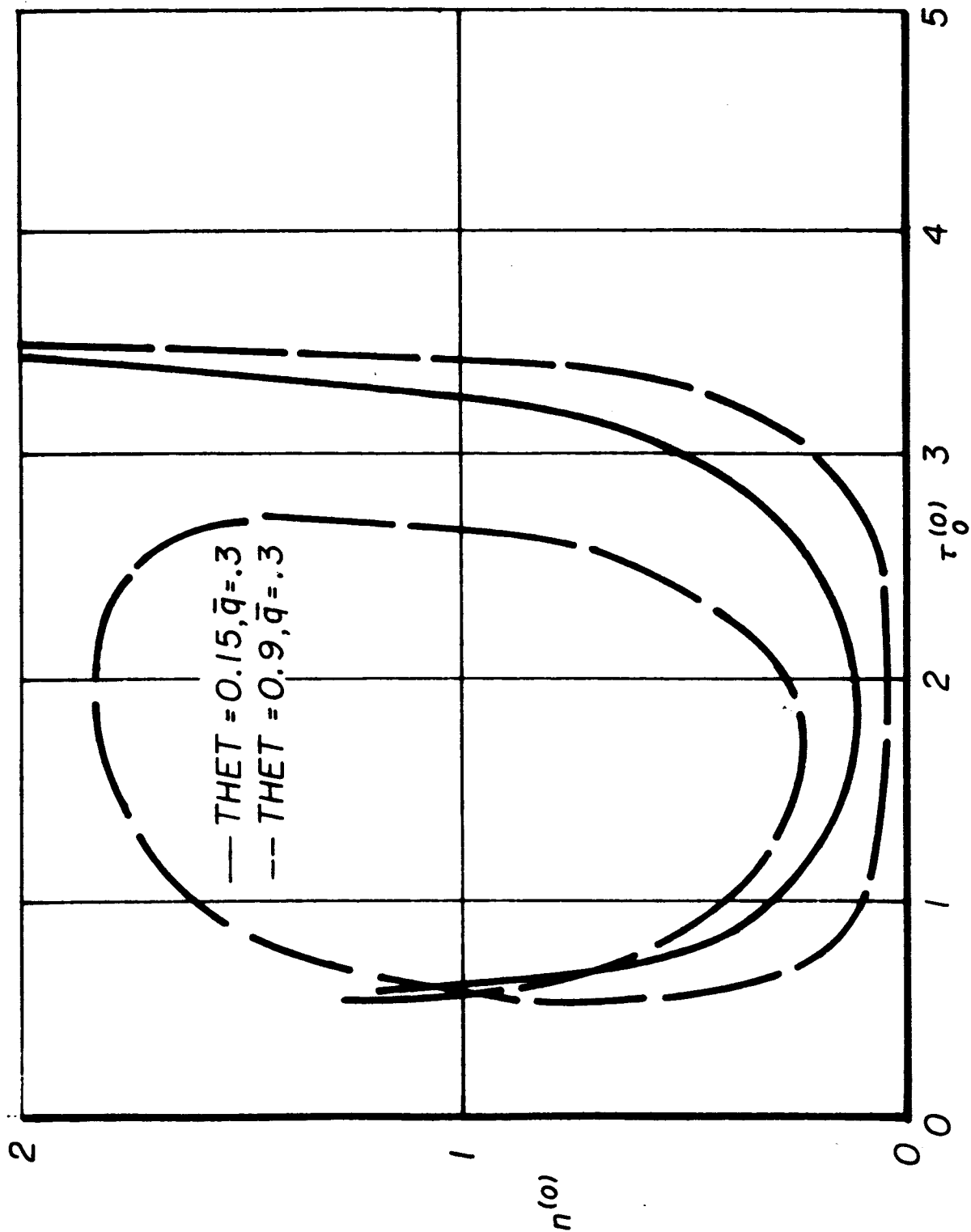


Figure 17

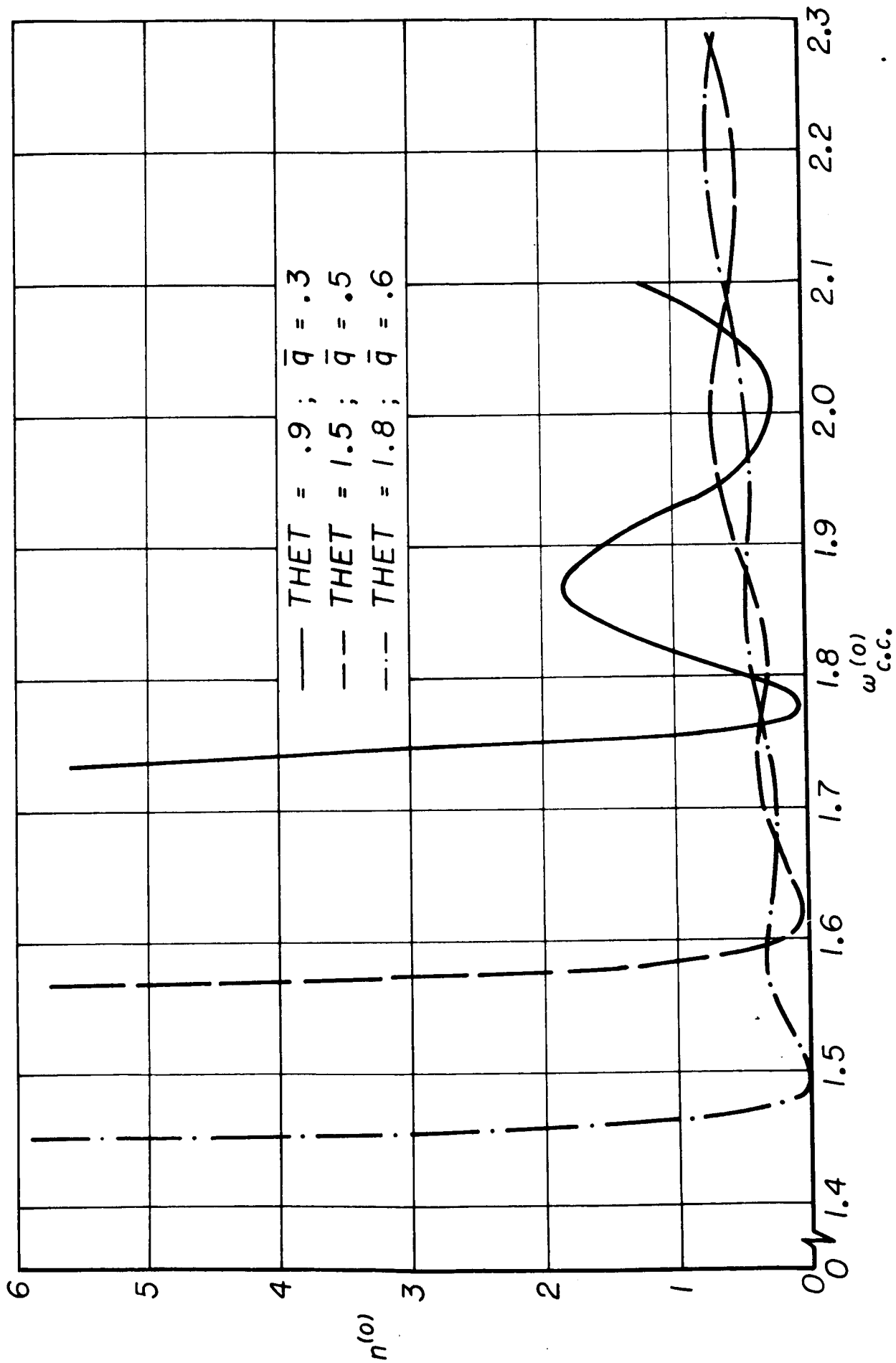
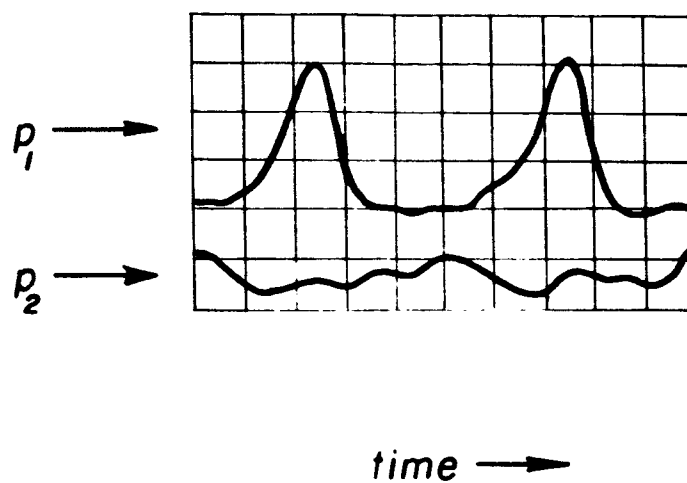
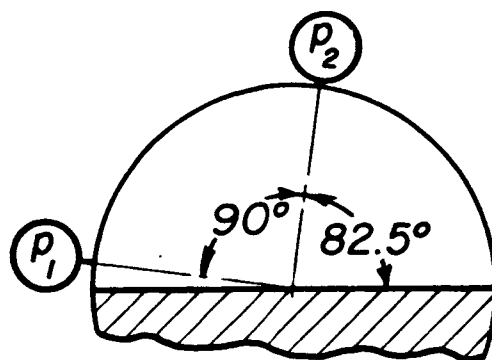
Typical solutions of $n^{(0)}(\omega)$ 

Figure 18

*Standing mode wave shapes,
first tangential mode,
180° sector motor*



Pressure traces



Location of pressure transducers

Figure 19

Pressure wave form at injector face

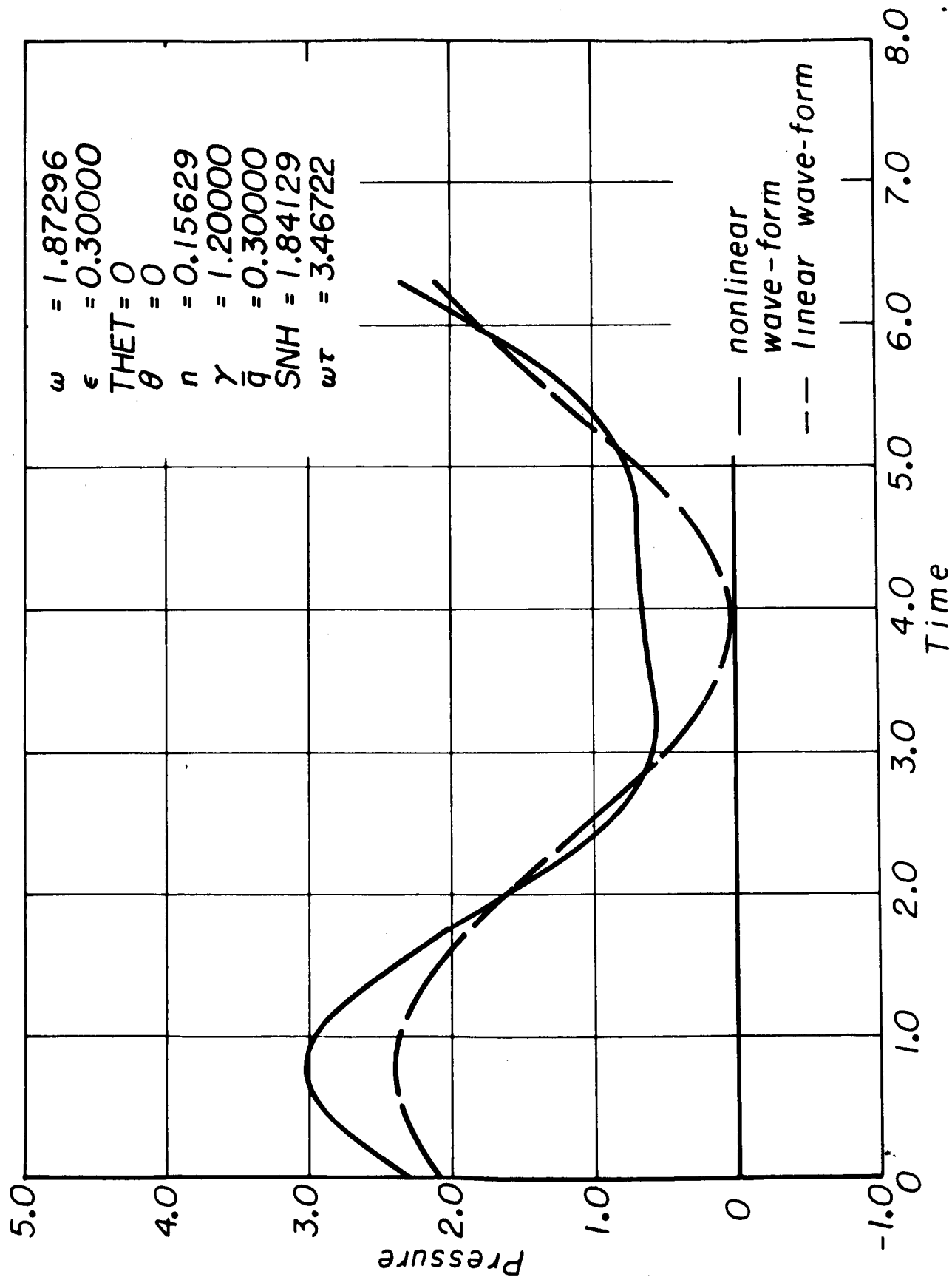


Figure 20

Comparison of pressure wave forms at the injector face and nozzle entrance

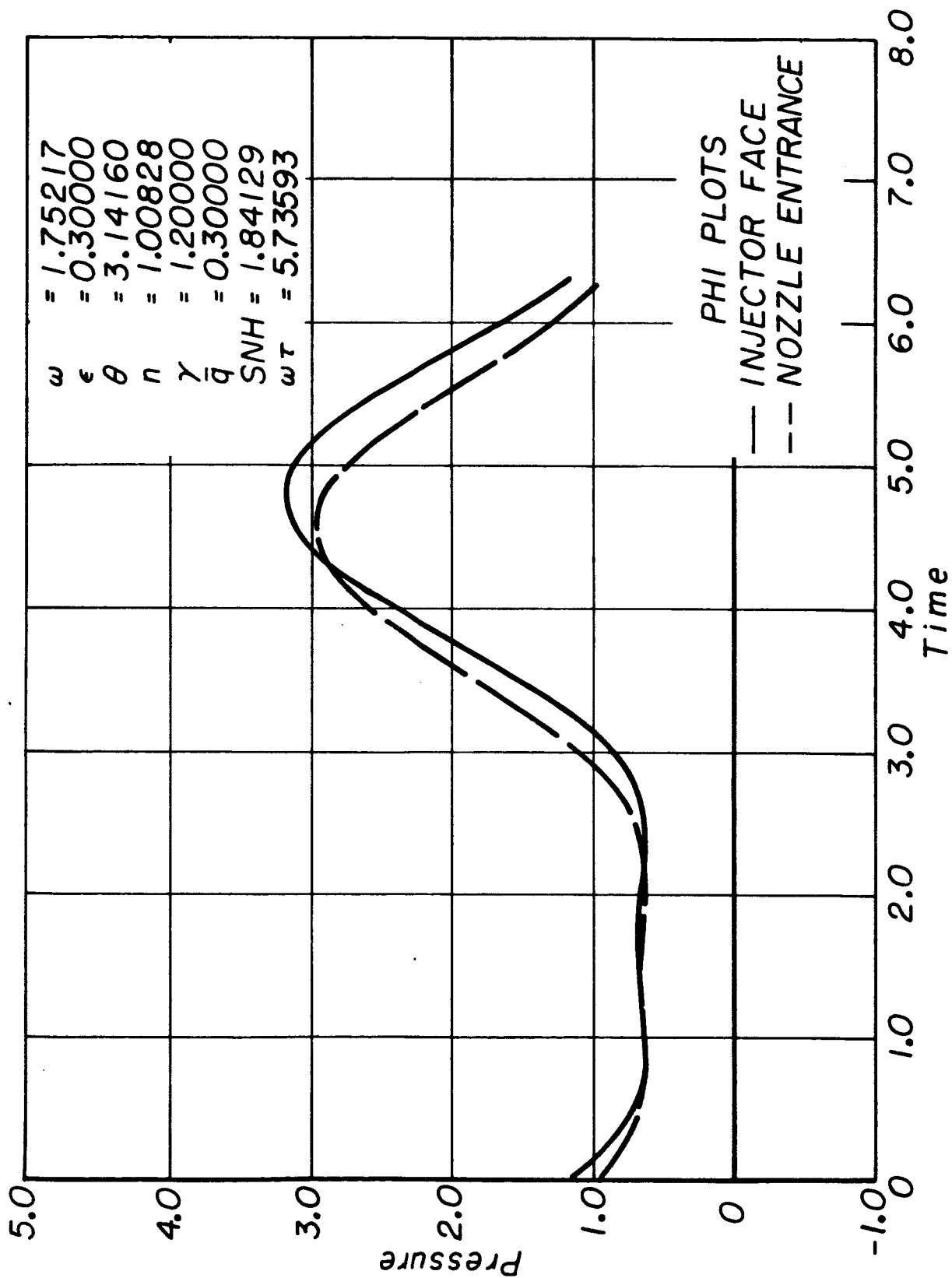


Figure 21

Comparison of pressure wave forms at the injector face and nozzle entrance

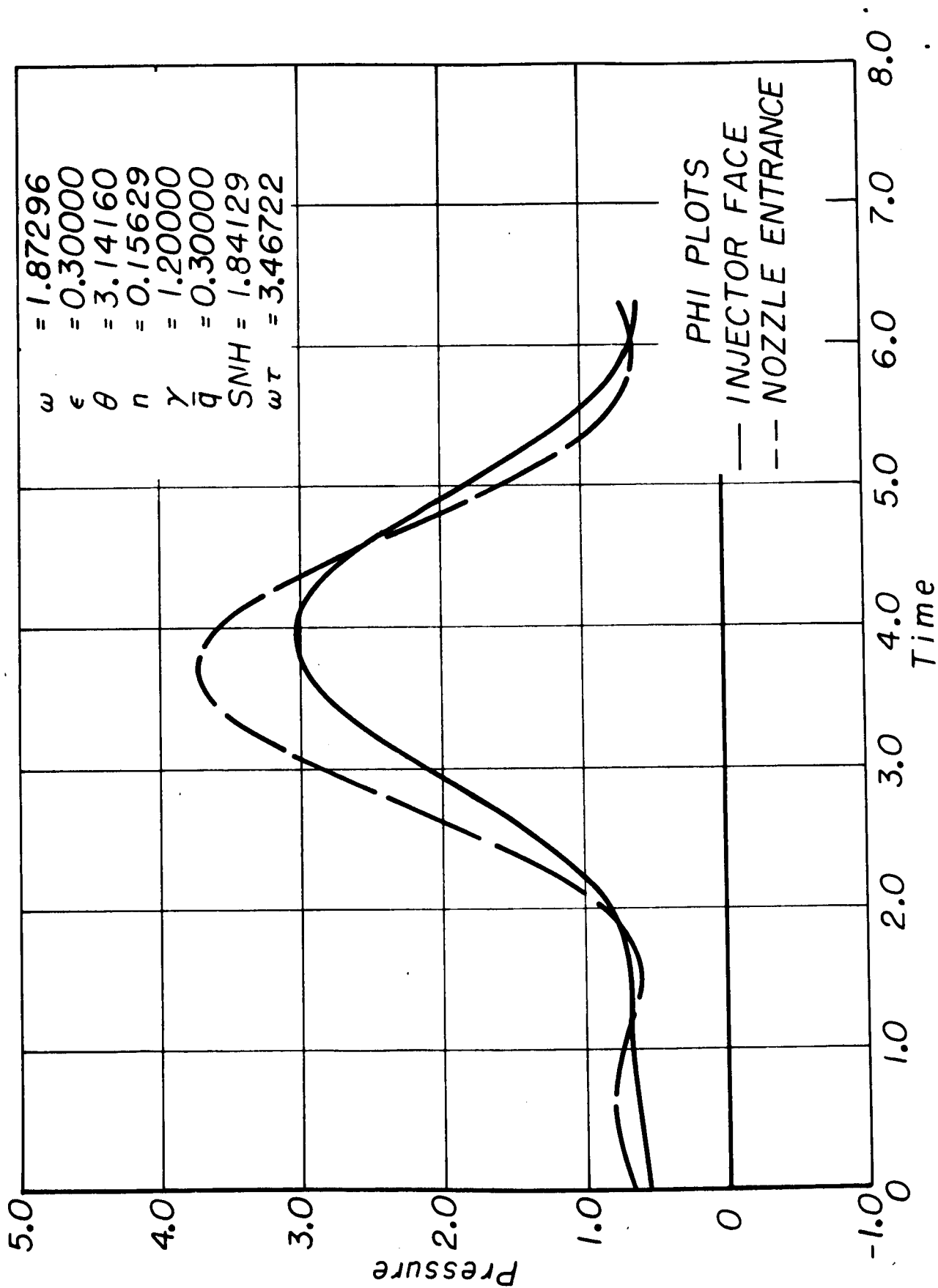


Figure 22

Comparison of pressure wave forms at the injector face and nozzle entrance

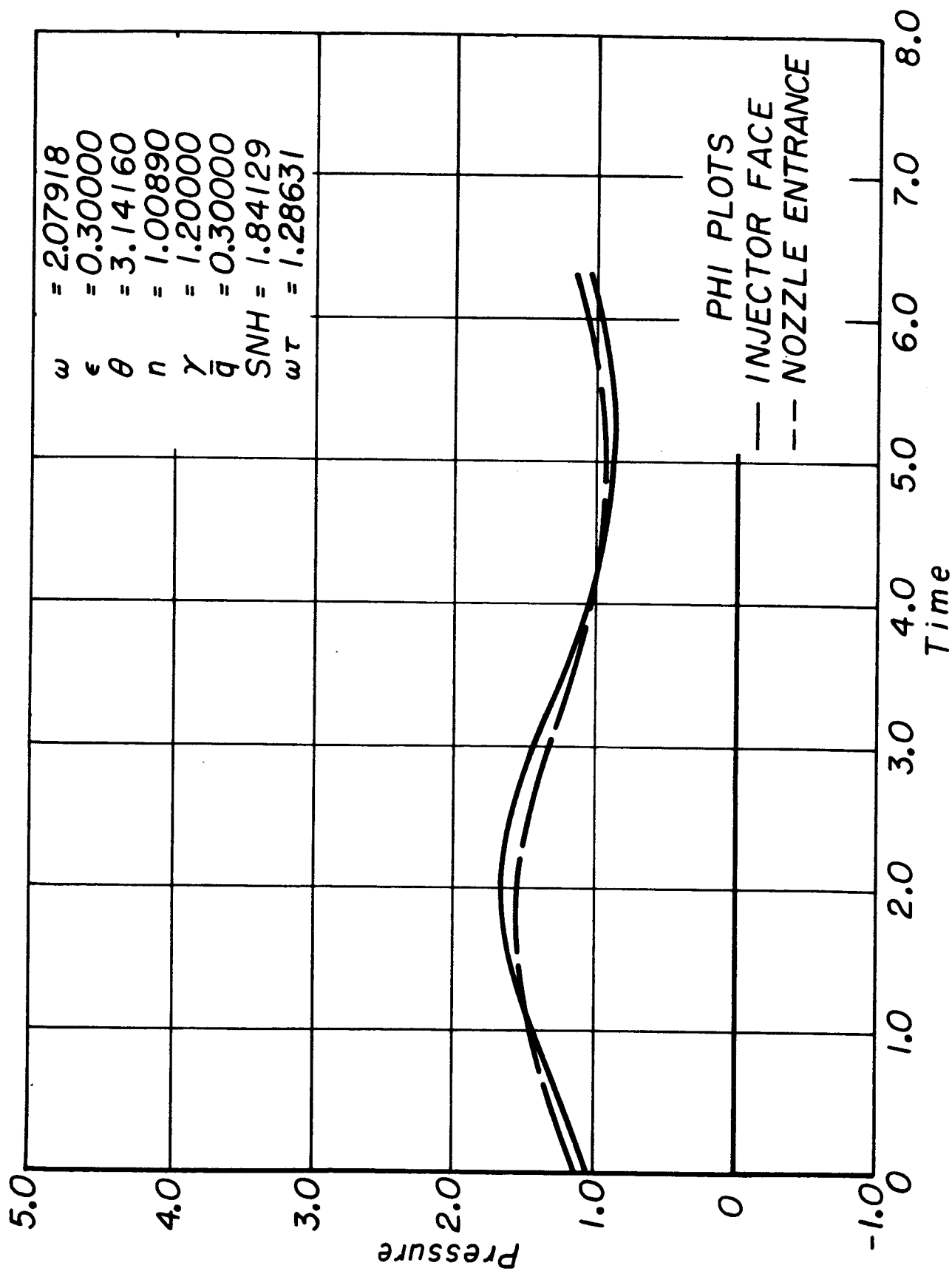


Figure 23

Comparison of pressure wave forms (at the injector face)
for different values of the frequency

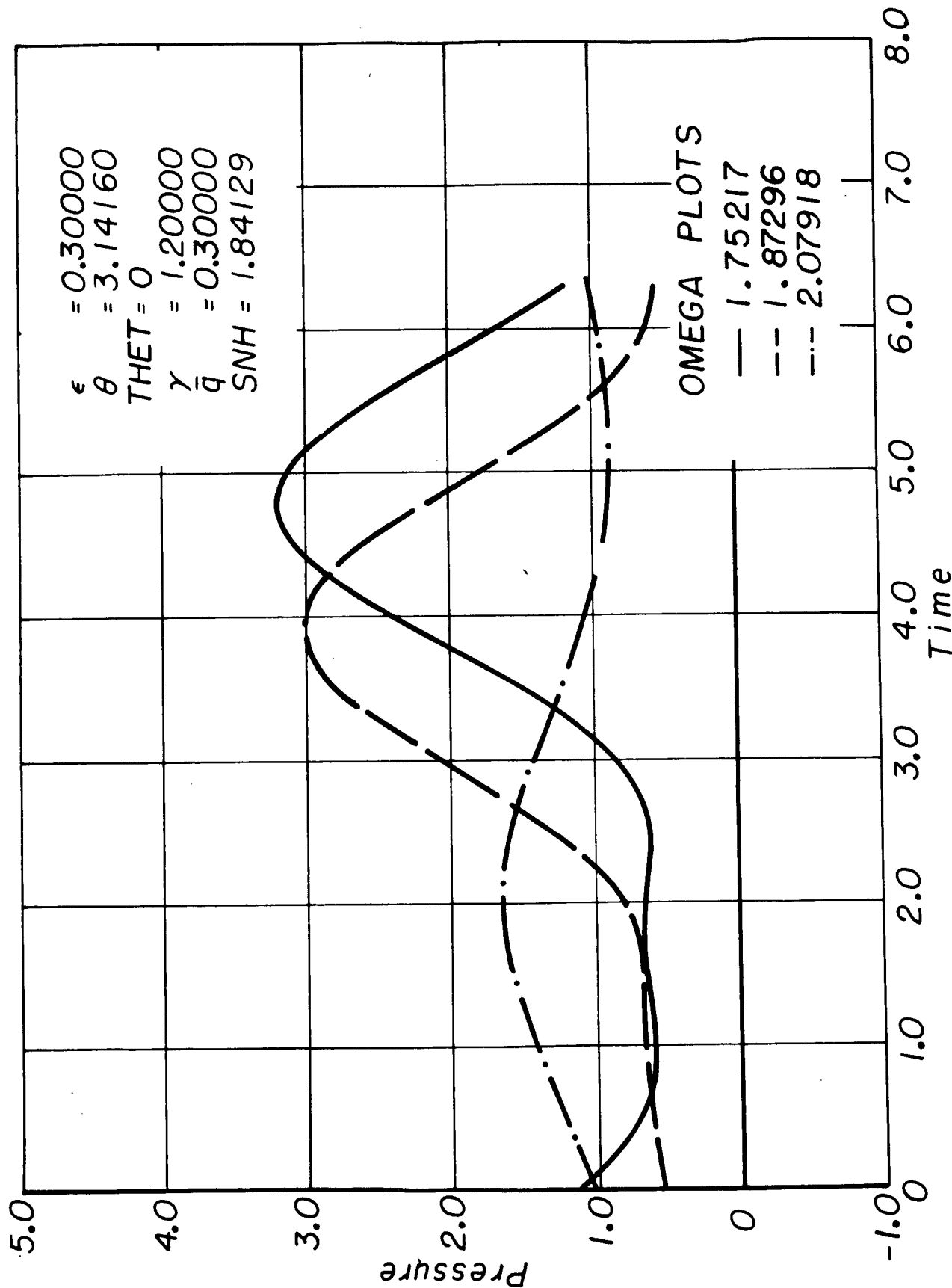


Figure 24

Comparison of pressure wave forms (at the nozzle entrance) for different values of the frequency

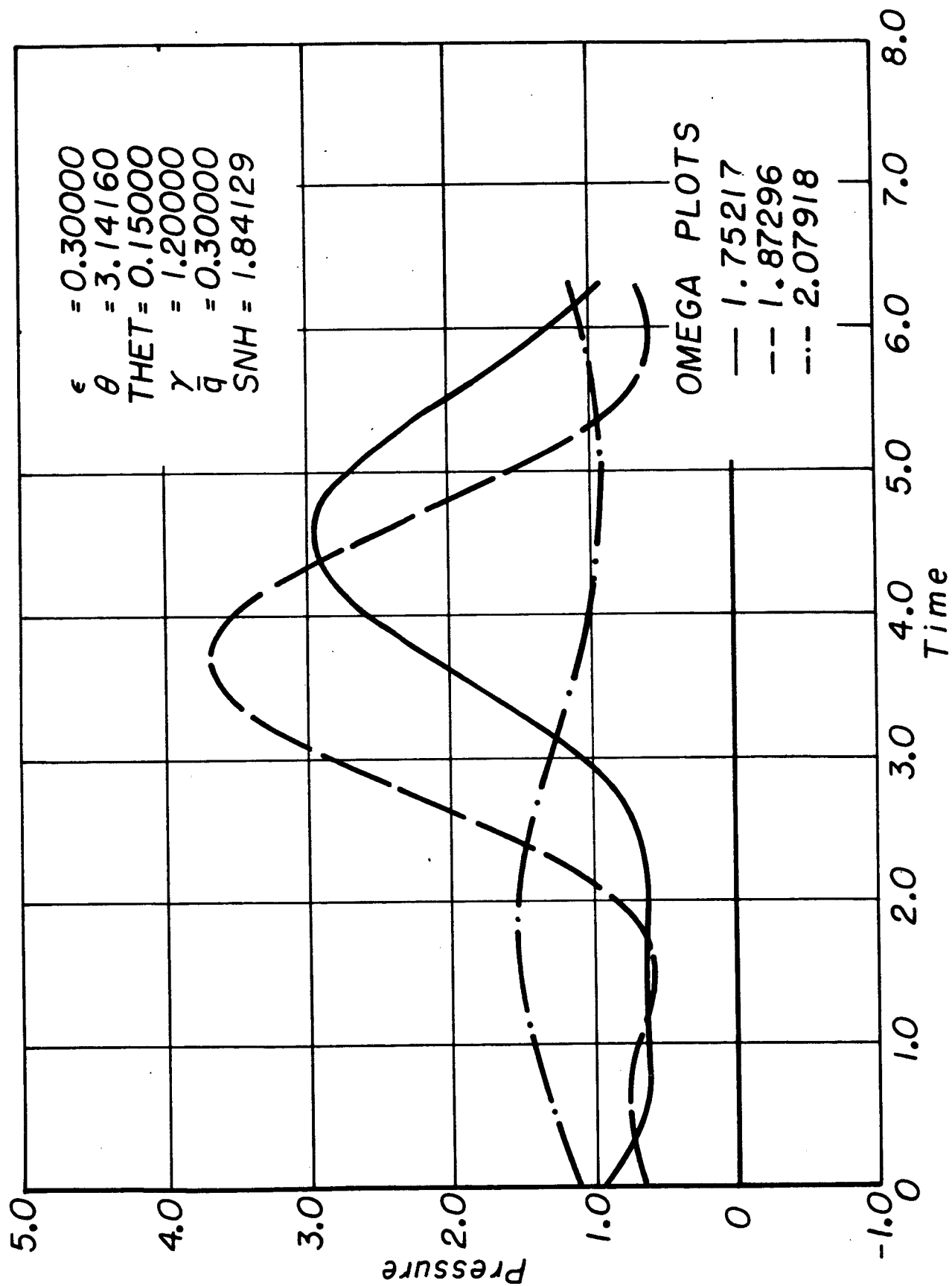


Figure 25

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Comparison of pressure wave forms for various transverse locations at the nozzle entrance

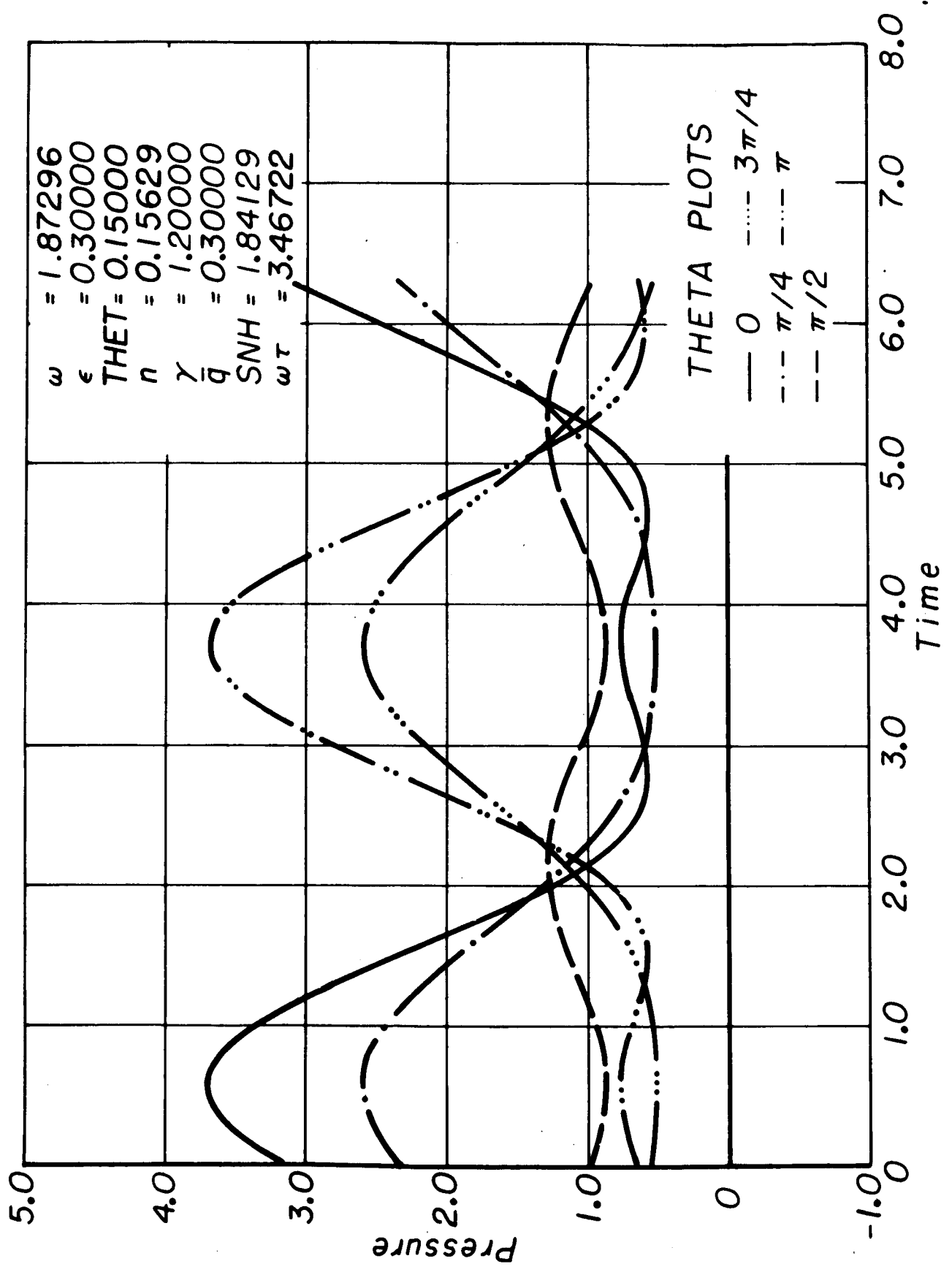


Figure 26

Comparison of pressure wave forms at the injector face and nozzle entrance

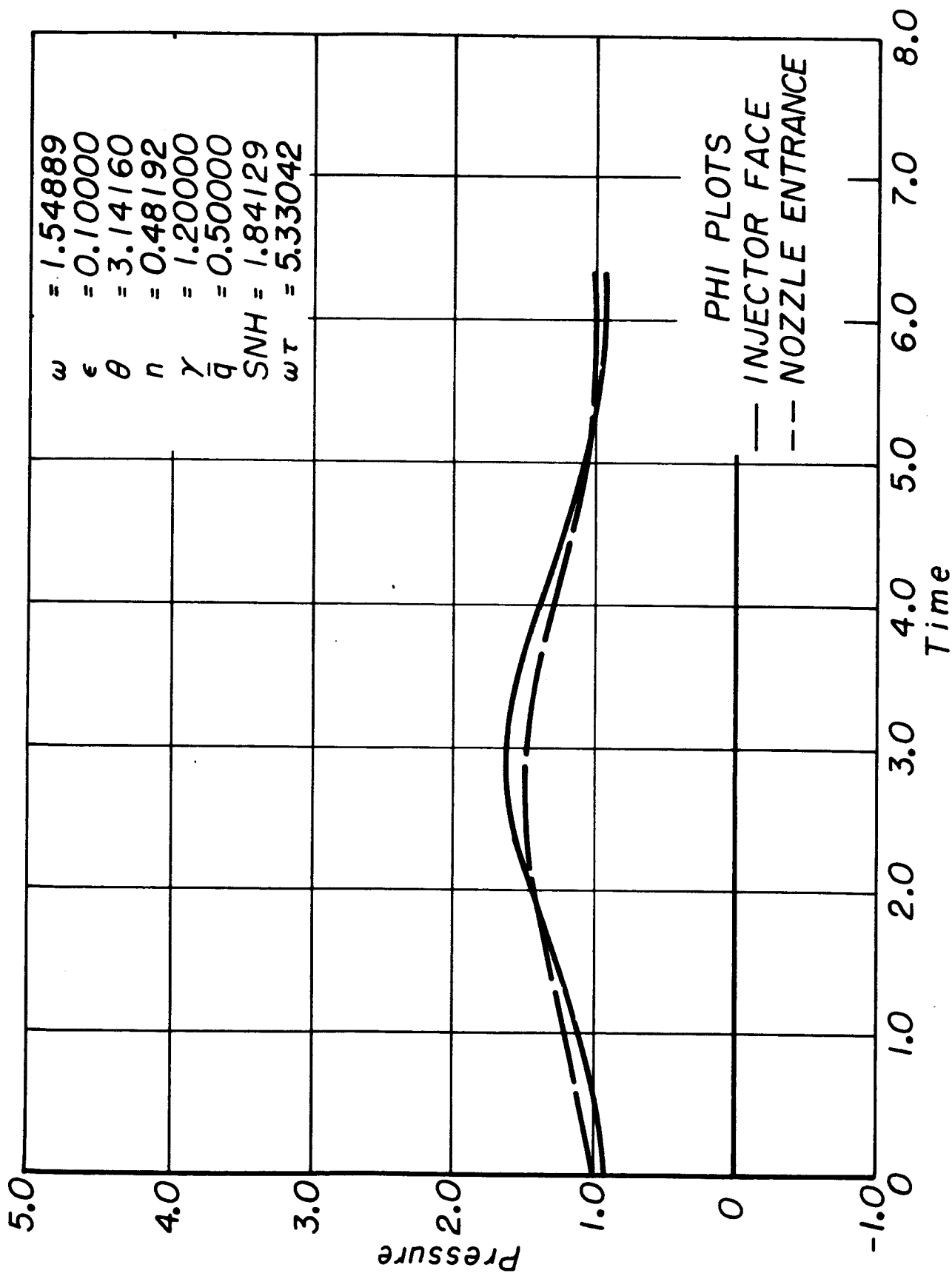


Figure 27

Comparison of pressure wave forms at the injector face and nozzle entrance

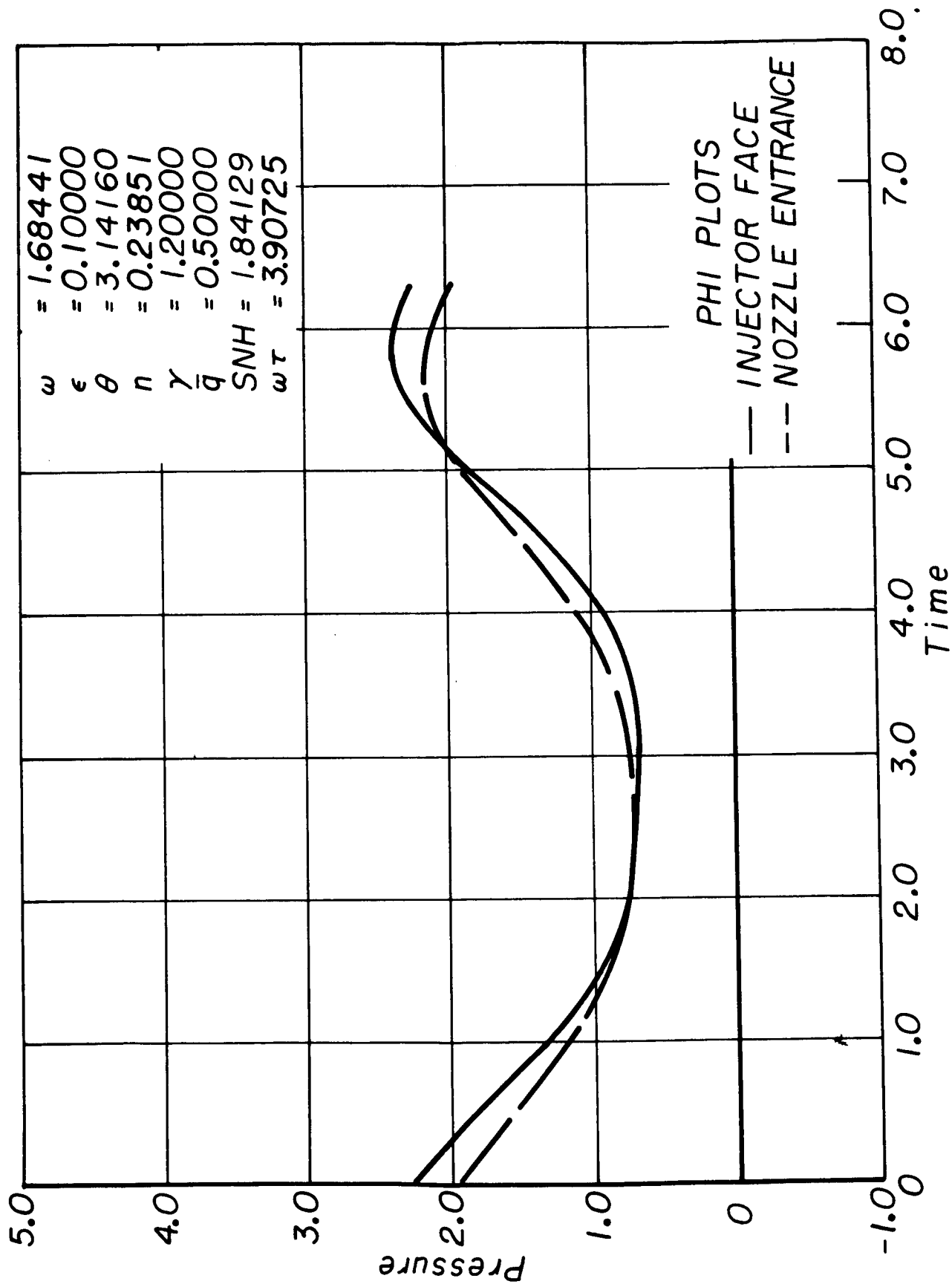


Figure 28

Comparison of pressure wave forms at the injector face and nozzle entrance

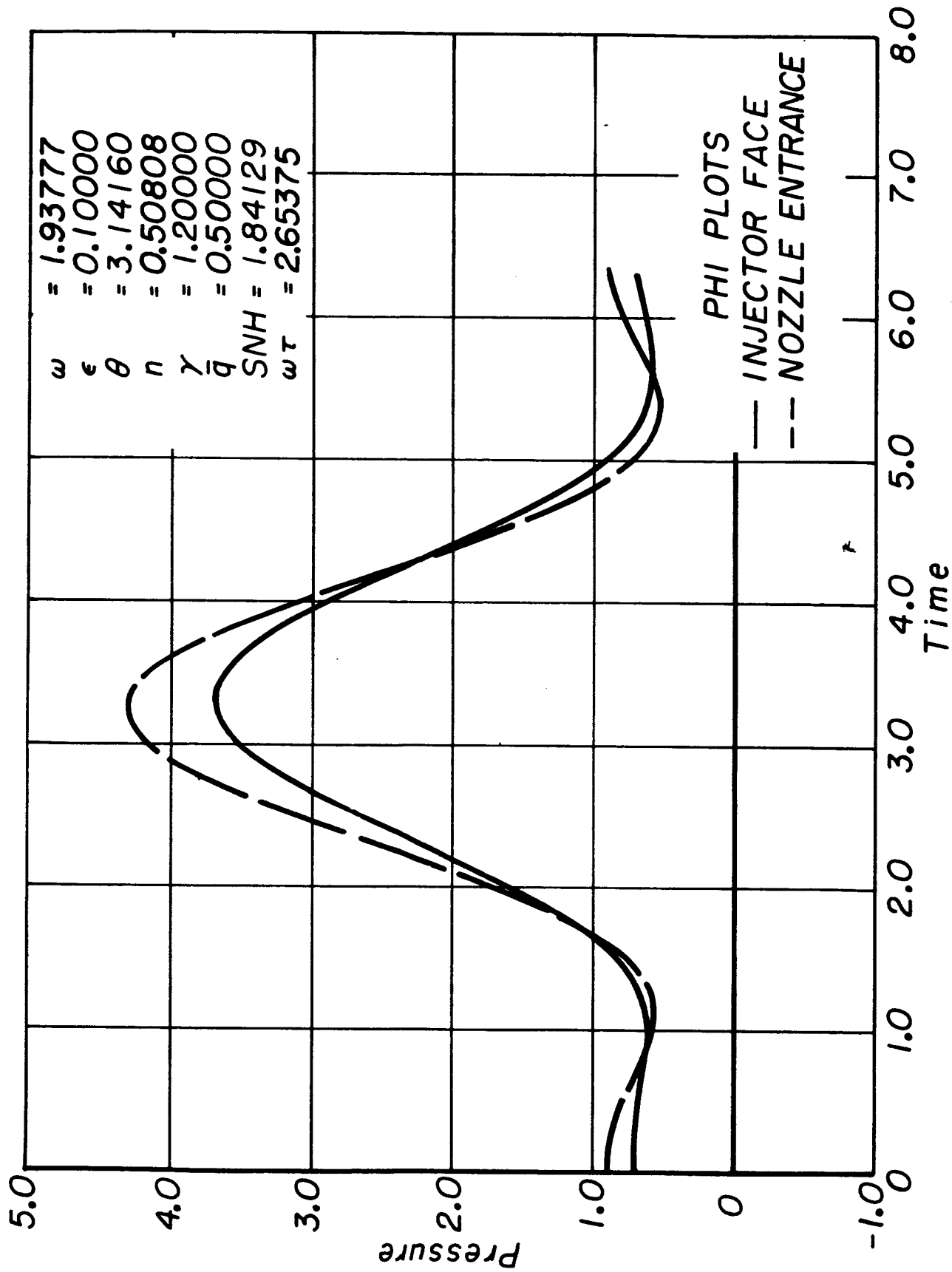


Figure 29

Comparison of pressure wave forms at the injector face and nozzle entrance

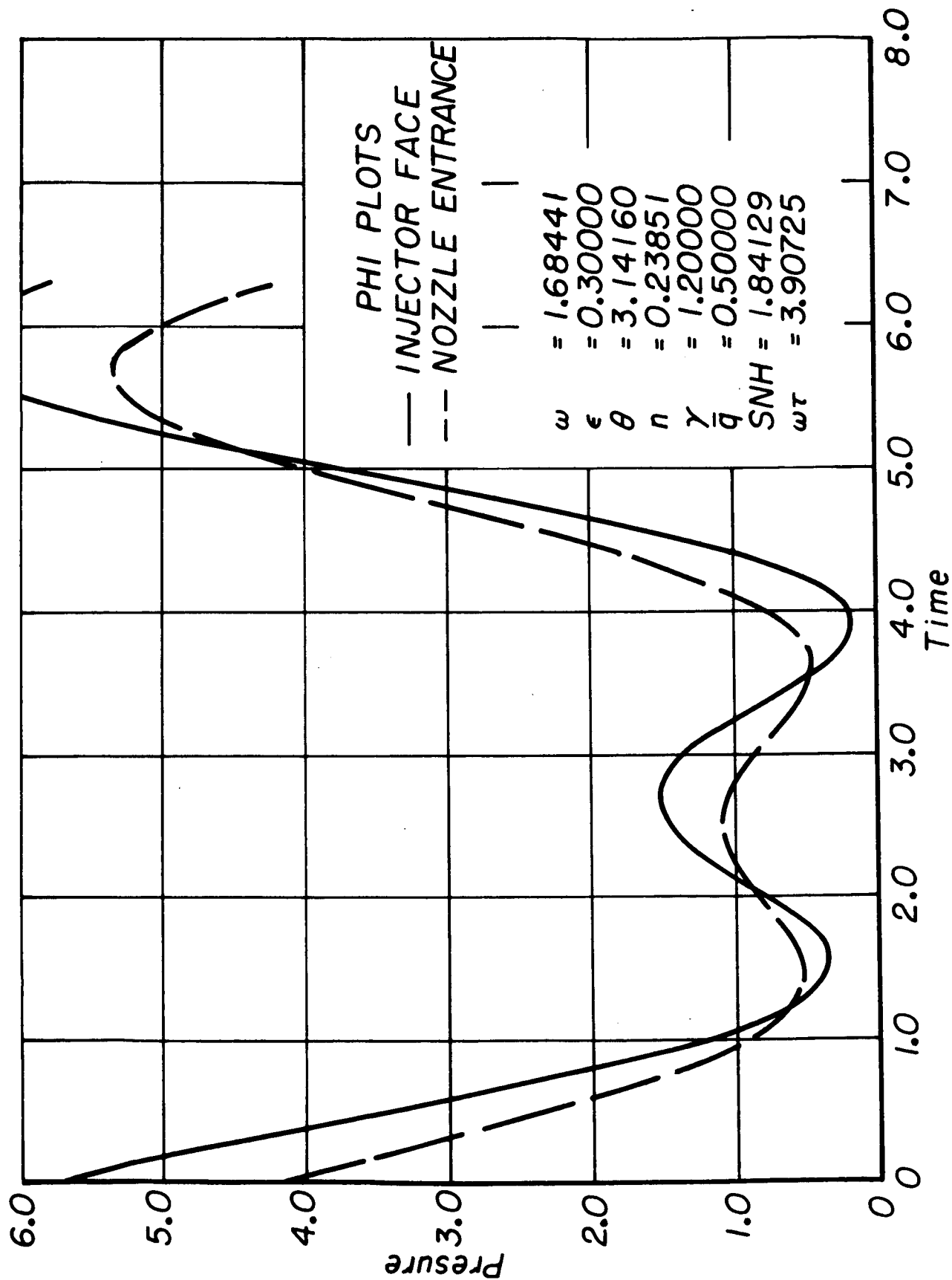


Figure 30

Comparison of pressure wave forms (at the injector face) for different values of the frequency

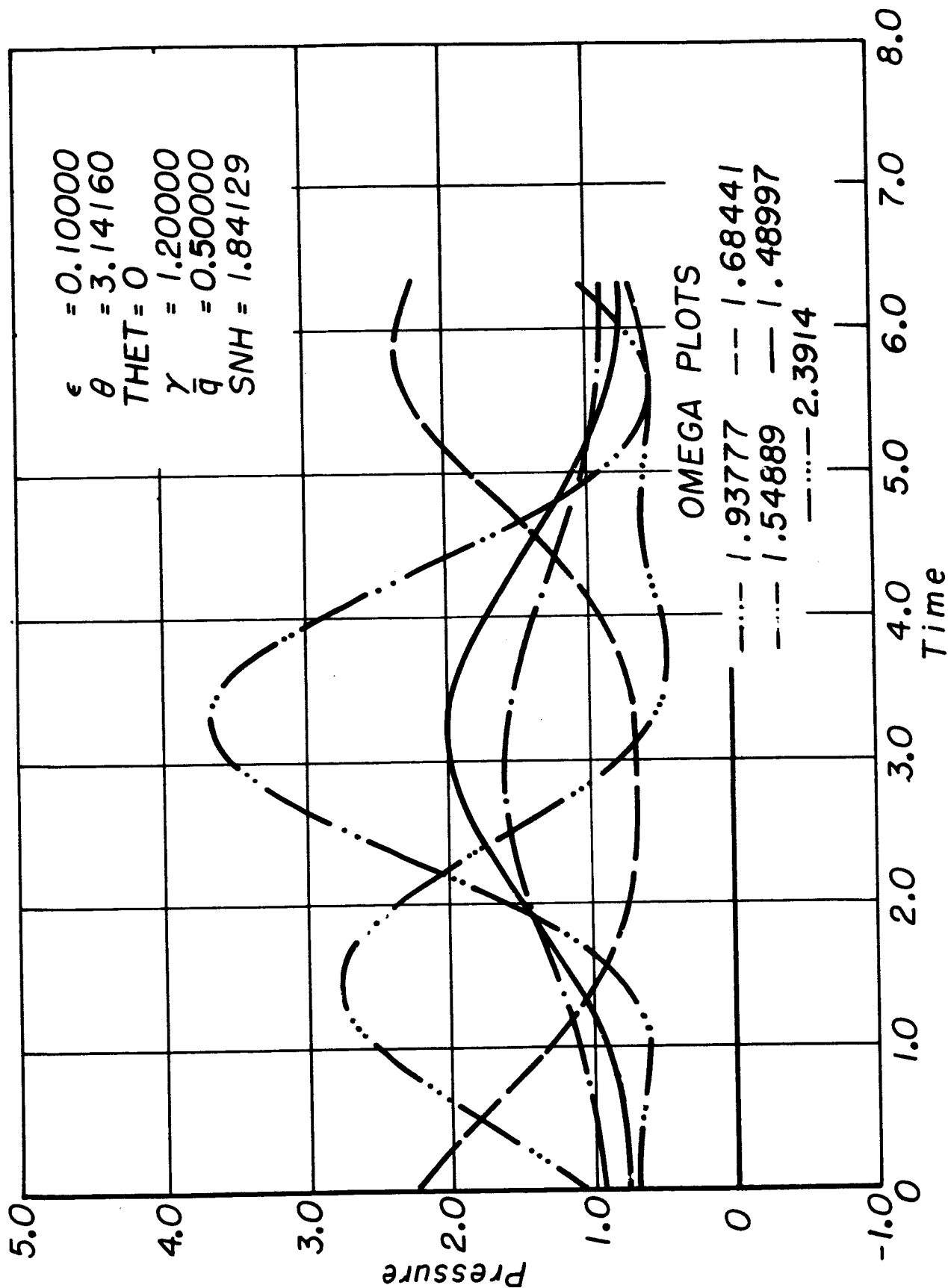


Figure 31

Comparison of pressure wave forms for various transverse locations
at the nozzle entrance

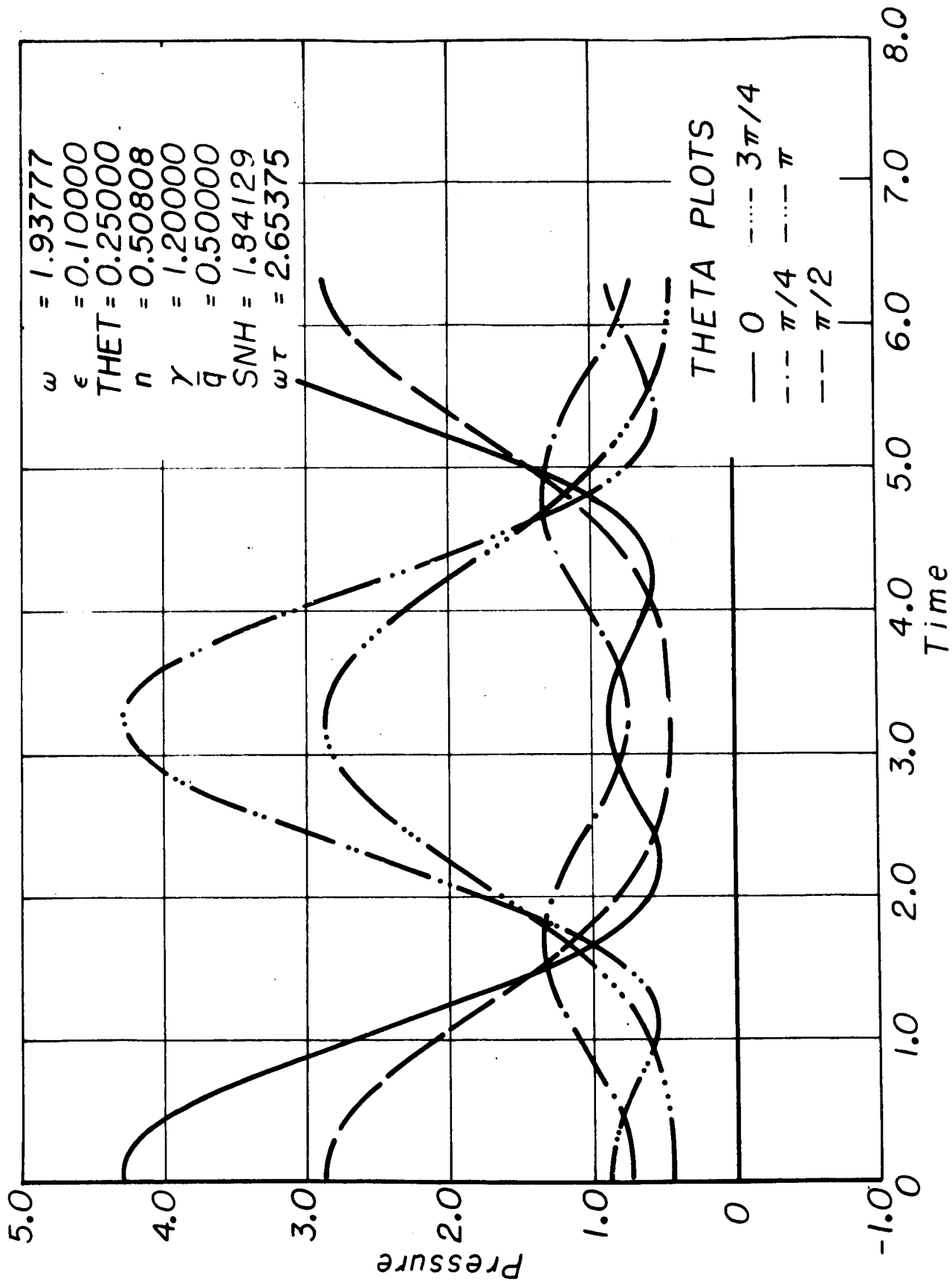


Figure 32

9-7 1.4 Tangential (1x12 Spuds) 500 lbs thrust 150 psi
pulsed instability limits tests

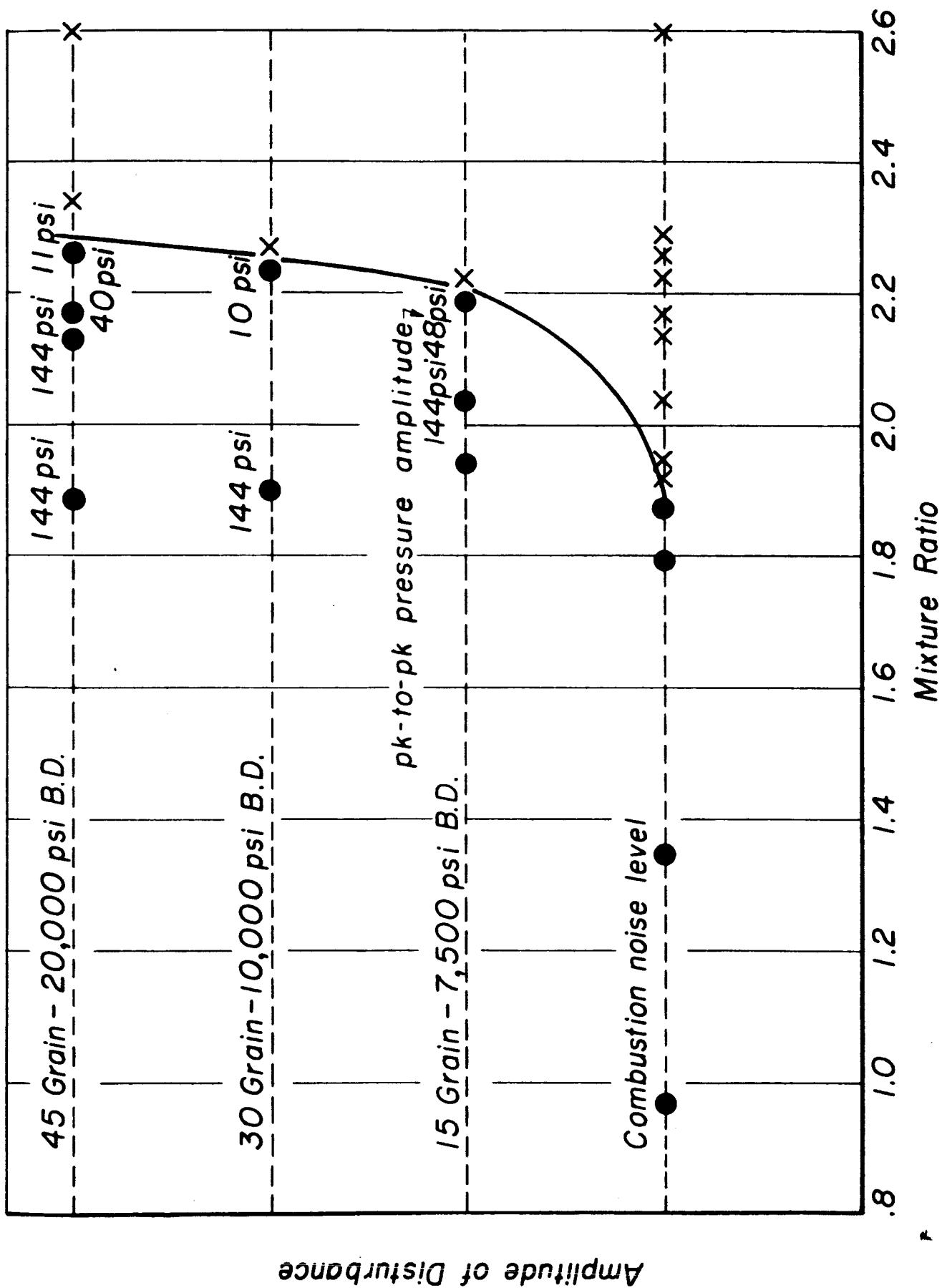


Figure 33

APPENDIX AUSE OF COMPLEX VARIABLESIN THE SOLUTION OF HIGHER ORDER EQUATIONS

Some of the methods employed in the solution of the equations encountered in this work will be demonstrated by use of a simple (and perhaps trivial) example.

Consider the following partial differential equation:

$$\frac{\partial y}{\partial x} - \lambda \frac{\partial y}{\partial t} = y^2 + y^3 \quad (\text{A-1})$$

where λ is an eigenvalue of the problem. The unknown y , which is assumed to represent a physically meaningful quantity, is a real variable which is known to be periodic in t . The following boundary condition must be satisfied by the solution of (A-1):

At $x = 1$ and for all values of t

$$y(1, t) = -\epsilon \sin t \quad \epsilon \ll 1 \quad (\text{A-2})$$

Equation (A-1) is a nonlinear partial differential equation. The nonlinearity of (A-1) and the presence of ϵ in the boundary conditions (A-2) suggest the use of perturbation technique for its solution. Accordingly it is assumed that y and λ have the following series representations.

$$y = y_1 \epsilon + y_2 \epsilon^2 + y_3 \epsilon^3 + O(\epsilon^4)$$

$$\lambda = \lambda_0 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + O(\epsilon^3) \quad (\text{A-3})^\#$$

The eigenvalue λ must be expanded in powers of ϵ in order to avoid the appearance of secular solutions.

Substituting (A-3) into (A-1) and separating the resulting equation according to powers of ϵ yields

$$\frac{\partial y_1}{\partial x} - \lambda_0 \frac{\partial y_1}{\partial t} = 0 \quad (\text{A-4})$$

$$\frac{\partial y_2}{\partial x} - \lambda_0 \frac{\partial y_2}{\partial t} = y_1^2 + \lambda_1 \frac{\partial y_1}{\partial t} \quad (\text{A-5})$$

$$\frac{\partial y_3}{\partial x} - \lambda_0 \frac{\partial y_3}{\partial t} = y_1^3 + 2y_1 y_2 + \lambda_2 \frac{\partial y_1}{\partial t} \quad (\text{A-6})^\#$$

Proceeding to solve (A-4), it is assumed that

$$y_1 = f_1(x)e^{it} \quad (\text{A-7})$$

where $f_1(x)$ may be a complex function of x and the desired solution (which represents a physical quantity) is the real part of (A-7). Note that the assumed form of y_1 satisfies the periodicity requirement in t . Substituting (A-7) into (A-4) gives

$$f_1' - i\lambda_0 f_1 = 0 \quad (\text{A-8})$$

which is a first order, linear homogeneous ordinary differential equation. Assuming that (A-8) has been solved and the appropriate boundary conditions satisfied, the analysis continues with the solution of the second order equation. Examination of (A-5) shows that it is an inhomogeneous, linear partial differential equation. All the variables appearing in (A-5) are real. Consequently with the available solution (which is complex) of y_1 the inhomogeneous part of (A-5) must be expressed in the following form:

$$y_1^2 = \left\{ \frac{1}{2} (y_1 + y_1^*) \right\}^2 = \frac{1}{4} (y_1^2 + 2y_1 y_1^* + y_1^{*2}) \quad (\text{A-9})$$

where an asterisk superscript $*$ indicates the complex conjugate of the

$\#$ It will be shown later that $\lambda_1 = 0$. Consequently all terms proportional to λ_1 were omitted from this equation.

considered variable. Most "second order equations",[#] encountered in mathematical physics, have several terms^{##} in their inhomogeneous parts. In these cases expressing the inhomogeneous parts of the higher order equations by the procedure outlined above (see Equation A-9) will result in very long and cumbersome equations. This difficulty can be partially overcome by the use of complex variables. Using complex notation, the inhomogeneous part of (A-5) is replaced by an equivalent complex expression, $F(x)$, such that

$$y_{1,r}^2 = \text{Re} \{ F(x) \} = \text{Re} \left\{ \frac{1}{2} (y_1^2 + y_1 y_1^*) \right\} \quad (\text{A-9a})^{###}$$

and the resulting solution y_2 is no longer real. The advantage of using complex notation becomes more apparent when the product of three different quantities, $A_r B_r C_r$, must be evaluated (this situation is often encountered in third order analysis). When A, B and C are available in complex form

[#] The phrase "second order equation" is used in quotation since it applies to the coefficient of ϵ^2 in the expansion of equation A-1 in terms of ϵ and not to a second order differential equation.

^{##} In most cases the terms appearing in the inhomogeneous part of "second order" equations will be products of two different first order quantities. When these quantities are available in their complex form, such a product will have to be written in the following form:

$$\begin{aligned} A_r B_r &= \frac{1}{2} (A + A^*) \frac{1}{2} (B + B^*) = \frac{1}{4} (AB + AB^* + A^* B + A^* B^*) \\ &= \text{Re} \left\{ \frac{1}{2} (AB + AB^*) \right\} = \text{Re} \left\{ \frac{1}{2} (AB + A^* B) \right\} \text{ etc.} \end{aligned}$$

^{###} When the inhomogeneous part contains a product of two different quantities the function $F(x)$ is given by the expressions in the above foot note.

the following relation holds:

$$\begin{aligned}
 A_r B_r C_r &= \frac{1}{2}(A + A^*) \frac{1}{2}(B + B^*) \frac{1}{2}(C + C^*) = \\
 &\frac{1}{8}(ABC + A^*BC + AB^*C + ABC^* + A^*B^*C^* + AB^*C^* + A^*BC^* \\
 &+ A^*B^*C) = \operatorname{Re} \left\{ \tilde{F}(x) \right\} = \operatorname{Re} \left\{ \frac{1}{4}(ABC + A^*BC + AB^*C + ABC^*) \right\}
 \end{aligned}
 \tag{A-10}$$

Consequently the use of the complex function \tilde{F} to represent $A_r B_r C_r$, in the inhomogeneous part of a third order equation will reduce the number of terms present there from eight to four.

Using complex variables and substituting Equations (A-9a) and (A-7) into (A-5) results in the following partial differential equation which describes the behavior of the second order solution:

$$\frac{\partial y_2}{\partial x} - \lambda_0 \frac{\partial y_2}{\partial t} = \frac{1}{2} f_1^2 e^{2it} + \frac{1}{2} f_1 f_1^* + i \lambda_1 f_1 e^{it}
 \tag{A-11}$$

As can be checked by inspection, the presence of $i \lambda_1 f_1 e^{it}$ in the inhomogeneous part of Equation (A-11) will result in the appearance of a secular component in the second order solution. Since in the present investigation we are interested in finding periodic solutions only, the appearance of secular solutions must be avoided. This can be done by requiring that $\lambda_1 = 0$. Imposing this condition, the second order solution can be expressed in the following form:

$$y_2 = f_2^{(2)}(x) e^{2it} + f_2^{(0)}(x)
 \tag{A-12}$$

and substituting (A-12) into (A-11) gives

$$f_2^{(2)'} - 2i \lambda_0 f_2^{(2)} = \frac{1}{2} f_1^2
 \tag{A-13a}$$

$$f_2^{(0)'} = \frac{1}{2} f_1 f_1^*
 \tag{A-13b}$$

The solution of (A-13) and the application of the proper boundary conditions will provide the solution of y_2 . The form of the "guessed" solution, i.e., Equation (A-12), is actually determined by the form of the inhomogeneous part of (A-11). It can be also looked at as a simplified form of a Fourier-series (in terms of e^{imt}) where the coefficients of all the terms, with the exception of those appearing in (A-12) are zero.

Substituting (A-12) and (A-7) into (A-6) and following the procedure outlined above gives:

$$\begin{aligned} \frac{\partial y_3}{\partial x} - \lambda_0 \frac{\partial y_3}{\partial t} = & \left(\frac{1}{4} f_1^3 + \frac{1}{2} f_2^{(2)} f_1 \right) e^{3it} \\ & + \left(\frac{3}{4} f_1^2 f_1^* + f_1 (f_2^{(0)} + f_2^{(0)*}) \right) e^{it} + i\lambda_2 f_1 e^{it} \end{aligned} \quad (A-14)$$

Inspecting the forms of the first and second order solutions (i.e., Equations (A-22) and (A-23)) it can be shown that the following combination of terms

$$\left(\frac{3}{4} f_1^2 f_1^* + i\lambda_2 f_1 \right) e^{it} = \left(\frac{3}{4} + i\lambda_2 \right) f_1 e^{it} \quad (A-15)$$

which appears in the inhomogeneous part of Equation (A-14) satisfies the homogeneous portion of the same equation. It is a well known fact that the presence of such terms in the inhomogeneous part of a differential equation will result in the appearance of solutions which are secular in time. In the present example this difficulty can be avoided by simply requiring that

$$\lambda_2 = \frac{3}{4} i \quad (A-16)$$

The substitution of Equation (A-16) into Equation (A-14) will result in the elimination of the troublesome terms from its inhomogeneous part. In view of this substitution the solution of $y^{(3)}$ can be assumed to have the following form:

$$y_3 = f_3^{(3)} e^{3it} + f_3^{(1)} e^{it} \quad (A-17)$$

and substitution of (A-17) and (A-16) into (A-14) yields:

$$f_3^{(3)'} - 3i\lambda_0 f_3^{(3)} = \frac{1}{4} f_1^3 + \frac{1}{2} f_2^{(2)} f_1 \quad (\text{A-18})$$

$$f_3^{(1)'} - i\lambda_0 f_3^{(1)} = f_1 \left(f_2^{(0)} + f_2^{(0)*} \right) \quad (\text{A-19})$$

Before proceeding with the solution of the problem, the boundary condition (A-2) must be rewritten in complex notation. Since

$$-\epsilon \sin t = \operatorname{Re} \{ i\epsilon e^{it} \}$$

it will be required that at $x = 1$ and for all t

$$y(1, t) = i\epsilon e^{it} \quad \epsilon \ll 1 \quad (\text{A-20})$$

Substituting the assumed series solution of y (i.e., Equation (A-3)) into Equation (A-18) and separating the latter according to powers of ϵ yields the following boundary conditions:

$$\begin{aligned} y^{(1)}(1, t) &= i\epsilon e^{it} \\ y^{(2)}(1, t) &= 0 \\ y^{(3)}(1, t) &= 0 \end{aligned} \quad (\text{A-21})$$

which must be satisfied by the first, second and third order solutions.

As can be checked by inspection

$$\epsilon y^{(1)} = \epsilon e^{i(\lambda_0 x + t)} \quad (\text{A-22})$$

where $\lambda_0 = \frac{\pi}{2}$ is a solution which satisfies Equation (A-4) as well as the appropriate first order boundary condition. The parameter ϵ can be interpreted as the amplitude of the first order solution.

Once the form of the first order solution has been determined it is possible to proceed with the solution of the second order equations.

Solving Equations (A-13a) and (A-13b) and satisfying the appropriate second order boundary conditions yields the following second order solutions:

$$f_2^{(2)} = \frac{1}{2} e^{2i\lambda_0 x} (x - 1)$$

and

(A-23)

$$f_2^{(0)} = \frac{1}{2} (x - 1)$$

Using the available first and second order solutions and following the same procedures as used in the solution of the second order equations, the third order solution can be obtained without much difficulty. This straight forward procedure will not be given here.

The equations and boundary conditions encountered in the analysis of the combustion chamber flow are considerably more complicated than those considered in this appendix. In spite of this fact the techniques employed in the solution of the equations and in the determination of their eigenvalue perturbation are essentially the same in both cases.

In conclusion this example provides a simple illustration of the use of perturbation techniques, complex variables, and eigenfunction expansions in the solution of nonlinear equations. It also serves to demonstrate some of the basic ideas used in the determination of the eigenvalue perturbations. In this case as well as in the analysis of the combustion chamber flow, these perturbations are of $O(\epsilon^2)$.

APPENDIX BTHIRD ORDER NOTATION

The application of a perturbation technique to the solution of the equation describing the combustion chamber flow resulted in a third order partial differential equation which contained many terms in its inhomogeneous part. These terms are either products of first and second order quantities (which are expressed in eigenfunction expansions) or products of three first order quantities. The manipulation and eigenfunction expansion of these products will be demonstrated in this appendix.

Let $A^{(1)}$ and $B^{(2)}$ represent typical first and second order complex solutions.

$$A'' = A(\varphi) \Theta(\theta) \Psi(\psi) e^{i\gamma} \quad (\text{B-1})$$

where

$$\Theta(\theta) = \begin{matrix} \cos \nu\theta \\ \sin \nu\theta \end{matrix} \quad \begin{matrix} \text{for standing waves} \\ \end{matrix} \quad (\text{B-2})$$

$$\Theta(\theta) = e^{\pm i\nu\theta} \quad \text{for travelling waves}$$

$$\Psi(\psi) = J_\nu(S_{\nu,k}) \sqrt{\frac{\psi'}{\psi_w}} \quad (\text{B-3})$$

and

$$B^{(2)} = \sum_{g=1}^{\infty} \left\{ B_{(2m,2g)}^{(2)} e^{2i\gamma} + B_{(0,2g)}^{(2)} \right\} \cos 2\nu\theta J_{2\nu}(S_{2\nu,g}) \sqrt{\frac{\psi'}{\psi_w}} \\ + \sum_{g=0}^{\infty} \left\{ B_{(2m,g)}^{(2)} e^{2i\gamma} + B_{(0,g)}^{(2)} \right\} J_0(S_{0,g}) \sqrt{\frac{\psi'}{\psi_w}} \quad (\text{B-4})$$

for the standing wave solution.

$$B^{(2)} = \sum_{g=0}^{\infty} B_{(0,0,g)}^{(2)} J_0(S_{(0,g)} \sqrt{\frac{\psi'}{\psi_w}}) + \sum_{g=1}^{\infty} B_{(2m,2\mu,g)}^{(2)} e^{2i(\nu\theta + y)} J_{2\nu}(S_{(2\mu,g)} \sqrt{\frac{\psi'}{\psi_w}}) \quad (B-5)$$

for the travelling wave solution.

Considering the standing wave case the product $A_r^{(1)} B_r^{(2)}$ can be written in the following form:

$$\begin{aligned} A_r^{(1)} B_r^{(2)} &= \text{Re} \left\{ \frac{1}{2} B^{(2)} (A^{(1)} + A^{(1)*}) \right\} \\ &= \text{Re} \left\{ \frac{1}{2} \sum_{g=1}^{\infty} \left[(B_{(2m,2\mu,g)}^{(2)} e^{3iy} + B_{(0,2\mu,g)}^{(2)}) A^{(1)} \right. \right. \\ &\quad \left. \left. + (B_{(2m,2\mu,g)}^{(2)} e^{iy} + B_{(0,2\mu,g)}^{(2)} e^{-iy}) A^{(1)*} \right] \cos 2\nu\theta \cos \nu\theta J_{2\nu}(S_{(2\mu,g)} \sqrt{\frac{\psi'}{\psi_w}}) J_{\nu}(S_{(0,\mu,g)} \sqrt{\frac{\psi'}{\psi_w}}) \right. \\ &\quad \left. + \frac{1}{2} \sum_{g=0}^{\infty} \left[(B_{(2m,0,g)}^{(2)} e^{3iy} + B_{(0,0,g)}^{(2)}) A^{(1)} \right. \right. \\ &\quad \left. \left. + (B_{(2m,0,g)}^{(2)} e^{iy} + B_{(0,0,g)}^{(2)} e^{-iy}) \right] \cos \nu\theta J_0(S_{(0,g)} \sqrt{\frac{\psi'}{\psi_w}}) J_{\nu}(S_{(2\mu,g)} \sqrt{\frac{\psi'}{\psi_w}}) \right\} \end{aligned} \quad (B-6)$$

The products of Bessel functions appearing in (B-6) can be expanded in the following form:

$$\begin{aligned} J_{2\nu}(S_{(2\mu,g)} \sqrt{\frac{\psi'}{\psi_w}}) J_{\nu}(S_{(0,\mu,g)} \sqrt{\frac{\psi'}{\psi_w}}) &= \sum_{g'=1}^{\infty} A_{(2\mu,g')}^{(2\nu,g)} J_{\nu}(S_{(2\mu,g')} \sqrt{\frac{\psi'}{\psi_w}}) \\ &= \sum_{g'=1}^{\infty} A_{(2\mu,g')}^{(2\nu,g)} J_{3\nu}(S_{(2\mu,g')} \sqrt{\frac{\psi'}{\psi_w}}) \end{aligned} \quad (B-7a)$$

$$\begin{aligned}
J_0(S_{(0,g)} \sqrt{\frac{\Psi}{\Psi_w}}) J_\nu(S_{(\nu,h)} \sqrt{\frac{\Psi}{\Psi_w}}) &= \sum_{g'=1}^{\infty} A_{(\nu,g')}^{(0,g)} \bar{J}_\nu(S_{(\nu,g')} \sqrt{\frac{\Psi}{\Psi_w}}) \\
&= \sum_{g'=1}^{\infty} A_{(3\nu,g')}^{(0,g)} \bar{J}_{3\nu}(S_{(3\nu,g')} \sqrt{\frac{\Psi}{\Psi_w}})
\end{aligned} \tag{B-7b}$$

where

$$J'_\nu(S_{(\nu,g')}) = J'_{3\nu}(S_{(3\nu,g')}) = J'_{2\nu}(S_{(2\nu,g')}) = J'_0(S_{(0,g')}) = 0 \tag{B-8}$$

Substituting Equation (B-7) and the trigonometric identity

$$\cos 2\nu\theta \cos \nu\theta = \frac{1}{2} (\cos 3\nu\theta + \cos \nu\theta) \tag{B-9}$$

into Equation (B-6) and replacing the terms proportional to e^{-iy} by their complex conjugates gives:

$$\begin{aligned}
A_r^{(1)} B_r^{(2)} &= \text{Re} \left\{ \sum_{g=1}^{\infty} \sum_{g'=1}^{\infty} [A_{(3\nu,g')}^{(2\nu,g)} (C_{(3m,3\nu)}^{(2\nu,g)} e^{3iy} + C_{(m,3\nu)}^{(2\nu,g)} e^{iy}) \cos 3\nu\theta \bar{J}_{3\nu}(S_{(3\nu,g')} \sqrt{\frac{\Psi}{\Psi_w}}) \right. \\
&\quad \left. + (C_{(3m,\nu)}^{(2\nu,g)} A_{(\nu,g')}^{(2\nu,g)} e^{3iy} + C_{(m,\nu)}^{(2\nu,g)} A_{(\nu,g')}^{(2\nu,g)} e^{iy}) \cos \nu\theta \bar{J}_\nu(S_{(\nu,g')} \sqrt{\frac{\Psi}{\Psi_w}}) \right] \\
&\quad \left. + \sum_{g=0}^{\infty} \sum_{g'=1}^{\infty} (C_{(3m,\nu)}^{(0,g)} A_{(\nu,g')}^{(0,g)} e^{3iy} + C_{(m,\nu)}^{(0,g)} A_{(\nu,g')}^{(0,g)} e^{iy}) \cos \nu\theta \bar{J}_\nu(S_{(\nu,g')} \sqrt{\frac{\Psi}{\Psi_w}}) \right\}
\end{aligned} \tag{B-10}$$

where

$$C_{(3m, 3\nu)}^{(2\nu, g)} = \frac{1}{4} B_{(2m, 2\nu, g)}^{(2)} A^{(1)} \quad (B-11)$$

$$C_{(m, 3\nu)}^{(2\nu, g)} = \frac{1}{4} (B_{(2m, 2\nu, g)}^{(2)} A^{(1)*} + A^{(1)} (B_{(0, 2\nu, g)}^{(2)} + B_{(0, 2\nu, g)}^{(2)*})) \quad (B-12)$$

$$C_{(3m, \nu)}^{(2\nu, g)} = C_{(3m, 3\nu)}^{(2\nu, g)} \quad (B-13)$$

$$C_{(3m, \nu)}^{(0, g)} = \frac{1}{2} B_{(2m, 0, g)}^{(2)} A^{(1)} \quad (B-14)$$

$$C_{(m, \nu)}^{(2\nu, g)} = C_{(m, 3\nu)}^{(2\nu, g)} \quad (B-15)$$

$$C_{(m, \nu)}^{(0, g)} = \frac{1}{2} (B_{(1m, 0, g)}^{(2)} + A^{(1)} (B_{(0, 0, g)}^{(2)} + B_{(0, 0, g)}^{(2)*})) \quad (B-16)$$

The final expression to be used in the inhomogeneous part of the third order expressions is obtained by interchanging the order of summation of Equation (B-10).

When travelling wave solutions are being considered, Equation (B-5) must be used. Multiplying (B-5) by the appropriate form of (B-1) and repeating the steps that led to the derivation of Equation (B-10) result in

the following expression:

$$\begin{aligned}
 A_r^{(1)} B_r^{(2)} = & \sum_{q'=1}^{\infty} \left\{ \frac{1}{2} \sum_{q=1}^{\infty} \left[A_{(3,q')}^{(2,q)} B_{(2m,2q,q')}^{(2)} A^{(1)} e^{3i(\nu\theta+y)} J_{3\nu}(S_{(3,q')} \sqrt{\frac{\psi}{\psi_0}}) \right. \right. \\
 & + A_{(\nu,q')}^{(2,q)} B_{(2m,2q,q')}^{(2)} A^{(1)*} e^{i(\nu\theta+y)} J_{\nu}(S_{(\nu,q')} \sqrt{\frac{\psi}{\psi_0}}) \left. \right] \\
 & + \frac{1}{2} \sum_{q=0}^{\infty} A_{(\nu,q')}^{(1,q)} (B_{(1,0,q')}^{(2)} + B_{(1,0,q')}^{(2)*}) e^{i(\nu\theta+y)} J_{\nu}(S_{(\nu,q')} \sqrt{\frac{\psi}{\psi_0}}) \left. \right\}
 \end{aligned}
 \tag{B-17}$$

to be used in the inhomogeneous part of the third order equation describing the behavior of a travelling wave.

The product of three first order quantities (or a cube), which are available in complex form, can be represented in the following form:

$$A_r^{(1)} B_r^{(1)} C_r^{(1)} = R_e \left\{ \frac{1}{4} (ABC + A^*BC + AB^*C + ABC^*) \right\}
 \tag{B-18}$$

Assuming that A, B and C can be expressed in forms similar to that given by Equation (B-1), Expression (B-18) can then be rewritten in the following form:

$$A_r^{(1)} B_r^{(1)} C_r^{(1)} = R_e \left\{ \frac{1}{4} \tilde{A}(\varphi) e^{3iy} + \frac{1}{4} \tilde{B}(\varphi) e^{iy} \right\} \cos^3 \nu\theta J_{\nu}^3(S_{(\nu,h)} \sqrt{\frac{\psi}{\psi_0}})
 \tag{B-19}$$

where

$$\tilde{A}(\varphi) = A^{(1)}(\varphi) B^{(1)}(\varphi) C^{(1)}(\varphi)
 \tag{B-20}$$

$$\tilde{B}(\varphi) = A^{(1)*}(\varphi) B^{(1)}(\varphi) C^{(1)}(\varphi) + A^{(1)}(\varphi) B^{(1)*}(\varphi) C^{(1)}(\varphi) + A^{(1)}(\varphi) B^{(1)}(\varphi) C^{(1)*}(\varphi)
 \tag{B-21}$$

Using the trigonometric identity

$$\cos^3 \nu \theta = \frac{1}{4} \cos 3\nu \theta + \frac{3}{4} \cos \nu \theta \quad (\text{B-22})$$

and the expansions

$$J_\nu^3(s_{(\nu, h)} \sqrt{\frac{\psi'}{\psi_w}}) = \sum_{g'=1}^{\infty} A_{(\nu, g')}^{(14h)} J_{3\nu}(s_{(3\nu, g')} \sqrt{\frac{\psi'}{\psi_w}}) = \sum_{g'=1}^{\infty} A_{(\nu, g')}^{(14h)} \bar{J}_\nu(s_{(\nu, g')} \sqrt{\frac{\psi'}{\psi_w}}) \quad (\text{B-23})$$

in (B-19) gives:

$$\begin{aligned} A_r'' B_r'' C_r'' = \text{Re} \left\{ \sum_{g'=1}^{\infty} \frac{1}{16} \left[(\tilde{A}(\varphi) e^{3iy} + \tilde{B}(\varphi) e^{iy}) A_{(\nu, g')}^{(14h)} \cos 3\nu \theta \bar{J}_{3\nu}(s_{(3\nu, g')} \sqrt{\frac{\psi'}{\psi_w}}) \right. \right. \\ \left. \left. + 3(\tilde{A}(\varphi) e^{3iy} + \tilde{B}(\varphi) e^{iy}) A_{(\nu, g')}^{(14h)} \cos \nu \theta \bar{J}_\nu(s_{(\nu, g')} \sqrt{\frac{\psi'}{\psi_w}}) \right] \right\} \end{aligned} \quad (\text{B-24})$$

which is the appropriate expression to be used in the equations for standing waves. When travelling waves are analyzed, the following expression is derived from (B-19):

$$\begin{aligned} A_r'' B_r'' C_r'' = \text{Re} \left\{ \sum_{g'=1}^{\infty} \frac{1}{4} \left[A_{(\nu, g')}^{(14h)} \tilde{A}(\varphi) e^{3i(\nu\theta + y)} \bar{J}_{3\nu}(s_{(3\nu, g')} \sqrt{\frac{\psi'}{\psi_w}}) \right. \right. \\ \left. \left. + A_{(\nu, g')}^{(14h)} \tilde{B}(\varphi) e^{i(\nu\theta + y)} \bar{J}_\nu(s_{(\nu, g')} \sqrt{\frac{\psi'}{\psi_w}}) \right] \right\} \end{aligned} \quad (\text{B-25})$$

The products of terms encountered in the third order analysis of the nozzle and combustion chamber flow are similar in form to the products appearing in the examples of this appendix. With minor modifications, the manipulation of these terms follows the procedure used in the treatment of examples of this appendix.

APPENDIX CEXPANSION FORMS USED IN THE SOLUTIONOF HIGHER ORDER EQUATIONS

The following expansions were used in second order analysis:

$$J_\nu^2(S_{(0,q)} X) = \sum_{q=0}^{\infty} A_{(0,q)} \bar{J}_0(S_{(0,q)} X) \quad (C-1) \quad \#$$

$$= \sum_{q=1}^{\infty} A_{(2,q)} \bar{J}_{2\nu}(S_{(2,q)} X) \quad (C-2)$$

$$\frac{1}{4} S_{(0,q)}^2 (\bar{J}_\nu'(S_{(0,q)} X))^2 = \sum_{q=0}^{\infty} B_{(0,q)} \bar{J}_0(S_{(0,q)} X) \quad (C-3)$$

$$= \sum_{q=1}^{\infty} B_{(2,q)} \bar{J}_{2\nu}(S_{(2,q)} X) \quad (C-4)$$

$$\frac{1}{4X^2} J_\nu^2(S_{(0,q)}^2 X) = \sum_{q=0}^{\infty} C_{(0,q)} \bar{J}_0(S_{(0,q)} X) \quad (C-5)$$

$$= \sum_{q=1}^{\infty} C_{(2,q)} \bar{J}_{2\nu}(S_{(2,q)} X) \quad (C-6)$$

where

$$X = \left(\frac{\psi}{\psi_w} \right)^{1/2} \quad (C-7)$$

Note that when the functions are expanded in terms of $J_0(S_{(0,q)} X)$ the summation index q starts at zero. The summation index q , for the expansion in terms of Bessel functions of order higher than zero, starts at 1.

The Bessel functions appearing in Equations (C-1) through (C-6) satisfy the following relation:

$$J_0'(S_{(0,q)}) = J_{2\nu}'(S_{(2\nu,q)}) = J_\nu'(S_{(\nu,h)}) = 0 \quad (C-8)$$

The following products of Bessel functions appear in the third order equations:

$$J_\nu^3(S_{(\nu,h)} X) \quad (C-9a)$$

$$\frac{1}{4} S_{(\nu,h)}^2 J_\nu(S_{(\nu,h)} X) (J_\nu'(S_{(\nu,h)} X))^2 \quad (C-9b)$$

$$\frac{1}{X^2} J_\nu^3(S_{(\nu,h)} X) \quad (C-9c)$$

$$J_\nu(S_{(\nu,h)} X) J_0(S_{(0,q)} X) \quad (C-10a)$$

$$\frac{1}{4} S_{(0,q)} S_{(\nu,h)} J_0'(S_{(0,q)} X) J_\nu'(S_{(\nu,h)} X) \quad (C-10b)$$

$$\frac{1}{X^2} J_0(S_{(0,q)} X) J_\nu(S_{(\nu,h)} X) \quad (C-10c)$$

$$J_\nu(S_{(\nu,h)} X) J_{2\nu}(S_{(2\nu,q)} X) \quad (C-11a)$$

$$\frac{1}{4} S_{(2\nu,q)} S_{(\nu,h)} J_{2\nu}'(S_{(2\nu,q)} X) J_\nu'(S_{(\nu,h)} X) \quad (C-11b)$$

$$\frac{1}{X^2} J_{2\nu}(S_{(2\nu,q)} X) J_\nu(S_{(\nu,h)} X) \quad (C-11c)^\#$$

Each of the functions appearing in Equations (C-9a) through (C-11c) are expanded twice, once in terms of $J_3(S_{(3\nu,q)} X)$ and once in

The terms defined in Equations (C-10a) through (C-11c) result from the presence of products of first and second order quantities in the inhomogeneous part of the third order equation.

in terms of $J_\nu(S_{(\nu,q')})$ where the index q' goes from one to infinity.

The series representations of these functions are given in the following expressions:

$$J_\nu^3(S_{(\nu,h)} X) = \sum_{q'=1}^{\infty} N_{(\nu,q')}^{(\nu,\nu)} J_{3\nu}(S_{(3\nu,q')} X) \quad (C-12a)$$

$$= \sum_{q'=1}^{\infty} N_{(\nu,q')}^{(\nu,\nu)} J_\nu(S_{(\nu,q')} X) \quad (C-12b)$$

$$\frac{1}{4} S_{(\nu,h)}^2 J_\nu(S_{(\nu,h)} X) (J'_\nu(S_{(\nu,h)} X))^2 = \sum_{q'=1}^{\infty} D_{(\nu,q')}^{(\nu,\nu)} J_{3\nu}(S_{(3\nu,q')} X) \quad (C-13a)$$

$$= \sum_{q'=1}^{\infty} D_{(\nu,q')}^{(\nu,\nu)} J_\nu(S_{(\nu,q')} X) \quad (C-13b)$$

$$\frac{1}{X^2} J_\nu^3(S_{(\nu,h)} X) = \sum_{q'=1}^{\infty} M_{(\nu,q')}^{(\nu,\nu)} J_{3\nu}(S_{(3\nu,q')} X) \quad (C-14a)$$

$$= \sum_{q'=1}^{\infty} M_{(\nu,q')}^{(\nu,\nu)} J_\nu(S_{(\nu,q')} X) \quad (C-14b)$$

$$J_\nu(S_{(\nu,h)} X) J_0(S_{(0,g)} X) = \sum_{q'=1}^{\infty} A_{(\nu,q')}^{(0,g)} J_{3\nu}(S_{(3\nu,q')} X) \quad (C-15a)$$

$$= \sum_{q'=1}^{\infty} A_{(\nu,q')}^{(0,g)} J_\nu(S_{(\nu,q')} X) \quad (C-15b)$$

$$\frac{1}{4} S_{(0, \delta)} S_{(\mu, h)} J_0'(S_{(0, \delta)} X) J_\nu'(S_{(\mu, h)} X) = \sum_{\delta'=1}^{\infty} B_{(\mu, \delta')}^{(0, \delta)} J_{3\nu}(S_{(\mu, \delta')} X) \quad (\text{C-16a})$$

$$= \sum_{\delta'=1}^{\infty} B_{(\mu, \delta')}^{(0, \delta)} J_\nu(S_{(\mu, \delta')} X) \quad (\text{C-16b})$$

$$\frac{1}{X^2} J_0(S_{(0, \delta)} X) J_\nu(S_{(\mu, h)} X) = \sum_{\delta'=1}^{\infty} C_{(\mu, \delta')}^{(0, \delta)} J_{3\nu}(S_{(\mu, \delta')} X) \quad (\text{C-17a})$$

$$= \sum_{\delta'=1}^{\infty} C_{(\mu, \delta')}^{(0, \delta)} J_\nu(S_{(\mu, \delta')} X) \quad (\text{C-17b})$$

$$J_\nu(S_{(\mu, h)} X) J_{2\nu}(S_{(2, \delta)} X) = \sum_{\delta'=1}^{\infty} A_{(\mu, \delta')}^{(2, \delta)} J_{3\nu}(S_{(\mu, \delta')} X) \quad (\text{C-18a})$$

$$= \sum_{\delta'=1}^{\infty} A_{(\mu, \delta')}^{(2, \delta)} J_\nu(S_{(\mu, \delta')} X) \quad (\text{C-18b})$$

$$\frac{1}{4} S_{(\mu, \delta)} S_{(\nu, \delta)} J_\nu'(S_{(\mu, h)} X) J_{2\nu}'(S_{(2, \delta)} X) = \sum_{\delta'=1}^{\infty} B_{(\mu, \delta')}^{(\mu, \delta)} J_{3\nu}(S_{(\mu, \delta')} X) \quad (\text{C-19a})$$

$$= \sum_{\delta'=1}^{\infty} B_{(\mu, \delta')}^{(\mu, \delta)} J_\nu(S_{(\mu, \delta')} X) \quad (\text{C-19b})$$

$$\frac{1}{x^2} \bar{J}_{2\nu}(S_{(2\nu, q)} x) \bar{J}_{\nu}(S_{(\nu, q)} x) = \sum_{q=1}^{\infty} C_{(\nu, q)}^{(2\nu, q)} \bar{J}_{3\nu}(S_{(3\nu, q)} x) \quad (\text{C-20a})$$

$$= \sum_{q=1}^{\infty} C_{(\nu, q)}^{(2\nu, q)} J_{\nu}(S_{(\nu, q)} x) \quad (\text{C-20b})$$

The index q appearing on the left side of Equations (C-15a) through (C-20b) can take any of the values indicated in Equations (C-10) and (C-11). The above series are usually called the Fourier-Bessel expansion. Their properties as well as the formulae for the calculations of their coefficients are discussed in most texts of mathematical physics[#] and consequently will not be given here.

[#] See for example Reference 11.

APPENDIX D

DERIVATION OF THE EXPRESSIONS FOR THE
GRADIENT, DIVERGENCE AND ROTOR
IN (φ, ψ, θ) COORDINATE SYSTEM

The use of a (φ, ψ, θ) coordinate system is very helpful in studies of internal flow problems in which the boundaries of the conduit do not coincide with a constant value of any one of the coordinates in a "conventional" coordinate system (e.g., rectangular, cylindrical or spherical coordinate system). In this case, φ , a velocity potential function, designates the axial variation, ψ , the stream function designates the radial variation and θ designates the azimuthal variation. Using such a coordinate system and Equation (II-33) the square of an elementary length $d\ell$ can be written in the following form:

$$\begin{aligned} d\ell^2 &= h_\varphi^2 d\varphi^2 + h_\psi^2 d\psi^2 + h_\theta^2 d\theta^2 \\ &= \frac{1}{\bar{q}^2} d\varphi^2 + \left(\frac{1}{r\bar{f}\bar{q}}\right)^2 d\psi^2 + r^2 d\theta^2 \end{aligned} \quad (D-1)$$

from which we get

$$h_\varphi = \frac{1}{\bar{q}} ; \quad h_\psi = \frac{1}{r\bar{f}\bar{q}} ; \quad h_\theta = r \quad (D-2)$$

Using these relations the expressions for the gradients, divergences and rotors which appear in the equations describing the flow field will now be derived. Letting f be any scalar,

$$\vec{F} = F_\varphi \vec{e}_\varphi + F_\psi \vec{e}_\psi + F_\theta \vec{e}_\theta$$

any vector and \vec{e}_φ , \vec{e}_ψ , \vec{e}_θ the unit vectors which are respectively

perpendicular to the surfaces $\phi = \text{constant}$, $\psi = \text{constant}$ and $\theta = \text{constant}$ we can write.[#]

$$\begin{aligned}\nabla f &= \left(\frac{1}{h_\phi} \frac{\partial f}{\partial \phi} \right) \underline{e}_\phi + \left(\frac{1}{h_\psi} \frac{\partial f}{\partial \psi} \right) \underline{e}_\psi + \left(\frac{1}{h_\theta} \frac{\partial f}{\partial \theta} \right) \underline{e}_\theta \\ &= (\bar{\rho} \frac{\partial f}{\partial \phi}) \underline{e}_\phi + (r \bar{\rho} \bar{q} \frac{\partial f}{\partial \psi}) \underline{e}_\psi + \left(\frac{1}{r} \frac{\partial f}{\partial \theta} \right) \underline{e}_\theta\end{aligned}$$

(D-3)

$$\nabla \cdot \underline{F} = \frac{1}{h_\phi h_\psi h_\theta} \left(\frac{\partial}{\partial \phi} (h_\psi h_\theta F_\phi) + \frac{\partial}{\partial \psi} (h_\phi h_\theta F_\psi) + \frac{\partial}{\partial \theta} (h_\phi h_\psi F_\theta) \right)$$

$$= \bar{\rho} \bar{q}^2 \left(\frac{\partial}{\partial \phi} \left(\frac{F_\phi}{\bar{\rho} \bar{q}} \right) + \frac{\partial}{\partial \psi} \left(\frac{r F_\psi}{\bar{q}} \right) + \frac{\partial}{\partial \theta} \left(\frac{F_\theta}{r \bar{\rho} \bar{q}^2} \right) \right)$$

(D-4)

$$\nabla \times \underline{F} = \frac{1}{h_\phi h_\psi h_\theta} \begin{vmatrix} h_\phi \underline{e}_\phi & h_\psi \underline{e}_\psi & h_\theta \underline{e}_\theta \\ \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial \theta} \\ h_\phi F_\phi & h_\psi F_\psi & h_\theta F_\theta \end{vmatrix}$$

$$= \bar{\rho} \bar{q} \left(\frac{\partial}{\partial \psi} (r F_\theta) - \frac{\partial}{\partial \theta} \left(\frac{F_\psi}{r \bar{\rho} \bar{q}} \right) \right) \underline{e}_\phi$$

$$+ \frac{1}{r} \bar{q} \left(\frac{\partial}{\partial \theta} \left(\frac{1}{\bar{q}} F_\phi \right) - \frac{\partial}{\partial \phi} (r F_\theta) \right) \underline{e}_\psi$$

$$+ r \bar{\rho} \bar{q}^2 \left(\frac{\partial}{\partial \phi} \left(\frac{F_\psi}{r \bar{\rho} \bar{q}} \right) - \frac{\partial}{\partial \psi} \left(\frac{1}{\bar{q}} F_\phi \right) \right) \underline{e}_\theta$$

(D-5)

[#]

For a brief introduction on the derivation of these quantities see for example Appendix 7 in Reference 14.

APPENDIX EDERIVATION OF THE TANGENTIAL COMPONENTS OF THE VORTICITY

A slightly different procedure for the calculation of the transverse component of vorticity will be given in this appendix.

Subtracting Equation (II-46a) from Equation (II-47a) gives:

$$\begin{aligned} (\tilde{\eta}^{(j)} - F^{(j)})_{\varphi} + \frac{i k m \omega^{(j)}}{\bar{q}^2} (\tilde{\eta}^{(j)} - F^{(j)}) &= \frac{1}{\bar{q}^2} (B^{(j)} - A^{(j)} - \int_0^{\varphi} \frac{1}{z} \frac{d\bar{q}^2}{d\varphi} S^{(j)} d\varphi') \\ &= \frac{1}{\bar{q}^2} L^{(j)} \end{aligned}$$

where

$$L^{(j)} = B^{(j)} - A^{(j)} - \int_0^{\varphi} \frac{1}{z} \frac{d\bar{q}^2}{d\varphi} S^{(j)} d\varphi' \quad (\text{E-1})$$

Substitution of the eigenfunction expansions of $F^{(j)}$, $\tilde{\eta}^{(j)}$, $S^{(j)}$, $B^{(j)}$ and $A^{(j)}$ for $j = 2, 3$ into the above equations yields identical equations for each of the eigenfunctions. The solution of this equation can be written in the following form:

$$\begin{aligned} (\tilde{\eta}^{(j)} - F^{(j)})_{(km, n\nu, q)} &= (V_{(km, n\nu, q)}^{(j)} - \Phi_{(km, n\nu, q)}^{(j)}) \Theta_{n\nu}^{(j)} \Psi_{n\nu, q}^{(j)} \\ &= f_0^{(km)} \left(\int_0^{\varphi} \frac{L_{(km, n\nu, q)}^{(j)}}{\bar{q}^2 f_0^{(km)}} d\varphi' + \tilde{\eta}_{(km, n\nu, q)}^{(j)} - F_{(km, n\nu, q)}^{(j)} \right) \\ &= f_0^{(km)} \left(\int_0^{\varphi} \frac{L_{(km, n\nu, q)}^{(j)}}{\bar{q}^2 f_0^{(km)}} d\varphi' + V_{(km, n\nu, q)}^{(j)} - \Phi_{(km, n\nu, q)}^{(j)} \right) \Theta_{n\nu}^{(j)} \Psi_{n\nu, q}^{(j)} \end{aligned} \quad (\text{E-2})$$

where

$$V_{(km, n\nu, q)}^{(j)} - \Phi_{(km, n\nu, q)}^{(j)} = C_{1(km, n\nu, q)}^{(j)} \quad (\text{E-3})$$

using the definition of $\mathcal{L}^{(j)}$ the inhomogeneous part of Equation (E-2) can be rewritten in the following form:

$$\begin{aligned}
 I &= f_0^{(km)} \left\{ \int_0^\varphi \frac{B_{(km, n, q)}^{(j)}}{\bar{q}^2 f_0^{(km)}} - \frac{A_{(km, n, q)}^{(j)}}{f_0^{(km)}} d\varphi' - \int_0^\varphi \frac{1}{\bar{q}^2 f_0^{(km)}} \left(\int_0^{\varphi'} \frac{d\bar{q}^2}{d\varphi} f_0^{(km)} \left(\int_0^{\varphi''} \frac{D_{(km, n, q)}^{(j)}}{\bar{q}^2 f_0^{(km)}} d\varphi''' \right. \right. \right. \\
 &\quad \left. \left. + \sigma_{(km, n, q)}^{(j)} \right) d\varphi'' \right) d\varphi' + C_{1(km, n, q)}^{(j)} \left\{ \Theta_{n, q}^{(j)} \Psi_{n, q}^{(j)} \right. \\
 &= \left(-M_{(km, n, q)}^{(j)} - \sigma_{(km, n, q)}^{(j)} f_2^{(km)} + C_{1(km, n, q)}^{(j)} f_0^{(km)} \right) \Theta_{n, q}^{(j)} \bar{\Psi}_{n, q}^{(j)}
 \end{aligned}$$

(E-4)

where

$$\begin{aligned}
 -M_{(km, n, q)}^{(j)} &= f_0^{(km)} \left\{ \int_0^\varphi \frac{1}{\bar{q}^2 f_0^{(km)}} \left(B_{(km, n, q)}^{(j)} - A_{(km, n, q)}^{(j)} \right. \right. \\
 &\quad \left. \left. - \int_0^{\varphi'} \frac{1}{2} \frac{d\bar{q}^2}{d\varphi''} f_0^{(km)} \int_0^{\varphi''} \frac{D_{(km, n, q)}^{(j)}}{\bar{q}^2 f_0^{(km)}} d\varphi''' d\varphi'' \right) d\varphi' \right\}
 \end{aligned}$$

(E-5)

and $f_2^{(km)}$ is defined in Equation (II-79). Using the definitions of $F^{(j)}$ and partial differentiating Equation (E-2) with respect to φ and ψ gives the following expressions for the tangential components

of the vorticity:

$$\begin{aligned}
 \eta_{(km, n\nu, g)}^{(j)} - \Xi_{(km, n\nu, g)}^{(j)} &= \left(V_{(km, n\nu, g)}^{(j)} - U_{(km, n\nu, g)}^{(j)} \right) \Theta_{n\nu}^{(j)} \Psi_{n\nu, g}^{(j)} \\
 &= \left(f_0^{(km)} \right)' \left(\int_0^\varphi \frac{\mathcal{L}_{(km, n\nu, g)}^{(j)} d\varphi'}{\bar{g}^2 f_0^{(km)}} + C_{1(km, n\nu, g)}^{(j)} \right) \\
 &\quad + \frac{\mathcal{L}_{(km, n\nu, g)}^{(j)}}{\bar{g}^2} \Theta_{n\nu}^{(j)} \Psi_{n\nu, g}^{(j)}
 \end{aligned} \tag{E-6}$$

Examination of Equations (E-1) and (E-6) and using the results given in Equations (II-141) and (II-150)[#] it can be shown that

$$S_{(km, n\nu, q)}^{(j)}(\varphi) = 0$$

and

$$C_{1(km, n\nu, q)}^{(j)} = 0 \tag{E-7}$$

are necessary and sufficient conditions for the disappearance of the tangential component of the vorticity. Performing a similar analysis and assuming (as in the analysis of Chapter II) that the axial component of the vorticity is identically zero, it can be easily shown that satisfying the conditions stated in Equation (E-7) will also guarantee the disappearance of the radial component of vorticity.

[#] With the aid of Equations (II-141) and (II-150) it can be shown that for $j = 2, 3$ $\mathcal{L}_{(km, n\nu, q)}^{(j)} = 0$.

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Washington, D.C. 20360
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Defense Documentation Center Headquarters
Cameron Station, Building 5
5010 Duke Street
Alexandria, Virginia 22314
Attn: TISIA

Picatinny Arsenal
Dover, New Jersey 07801
Attn: E. Jenkins

Attn: Technical Librarian
Designee: I. Forsten, Chief
Liquid Propulsion Laboratory
SMUPA-DL

Redstone Scientific Information
Building 4484
Redstone Arsenal
Huntsville, Alabama
Attn: Technical Library

RTNT
Bolling Field
Washington, D.C. 20332
Attn: L. Green, Jr.

U.S. Army Missile Command
Redstone Arsenal
Alabama 35809
Attn: J. Connaughton

Attn: Technical Librarian
Designee: Walter Wharton

U.S. Atomic Energy Commission
Technical Information Services
Box 62
Oak Ridge, Tennessee
Attn: Technical Librarian
Designee: A.P. Huber
Gaseous Diffusion Plant
(ORGDP) P.O. Box P

U.S. Naval Ordnance Test Station
China Lake
California 93557
Attn: E.W. Price

Attn: Technical Librarian
Designee: Code 4562
Chief, Missile Propulsion Div.

CPIA

Chemical Propulsion Information Agency
Applied Physics Laboratory
The John Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland 20910
Attn: T.W. Christian

Attn: Technical Librarian
Designee: Neil Safeer

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Azusa, California 91703
Attn: Technical Librarian
Designee: L.F. Kohrs

Aerojet-General Corporation
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Sacramento, California 95809
Attn: R.J. Hefner
F.H. Reardon
Attn: Technical Librarian
Bldg. 2015, Dept. 2410
Designee: R. Stiff

Aeronutronic
Philco Corporation
Ford Road
Newport Beach, California 92663
Attn: Technical Librarian
Designee: D.A. Carrison

Aerospace Corporation
P.O. Box 95085
Los Angeles, California 90045
Attn: O.W. Dykema
W.C. Strahle

Attn: Technical Librarian
Designee: John G. Wilder
MS-2293
Propulsion Dept.

Astrosystems International, Inc.
1275 Bloomfield Avenue
Fairfield, New Jersey 07007
Attn: Technical Librarian
Designee: A. Mendenhall

Atlantic Research Corporation
Edsall Road and Shirley Highway
Alexandria, Virginia 22314
Attn: Technical Librarian
Designee: A. Scurlock

Battelle Memorial Institute
505 King Avenue
Columbus 1, Ohio
Attn: Charles E. Day,
Classified Rept. Librarian

Bell Aerosystems Company
P.O. Box 1
Buffalo 5, New York 14240
Attn: T. Rossman
J. Senneff

Attn: Technical Librarian
Designee: W.M. Smith

Boeing Company
P.O. Box 3707
Seattle, Washington 98124
Attn: Technical Librarian
Designee: J.D. Alexander

Chrysler Corporation
Missile Division
P.O. Box 2628
Detroit, Michigan 48231
Attn: Technical Librarian
Designee: John Gates

Curtiss-Wright Corporation
Wright Aeronautical Division
Wood-Ridge, New Jersey 07075
Attn: Technical Librarian
Designee: G. Kelley

Defense Research Corporation
6300 Hollister Avenue
P.O. Box 3587
Santa Barbara, California 93105
Attn: B. Gray
C.H. Yang

Douglas Aircraft Company
Missile & Space Systems Division
3000 Ocean Park Boulevard
Santa Monica, California 90406
Attn: Technical Librarian
Designee: R.W. Hallet
Advanced Space Tech.

Douglas Aircraft Company
Astropower Laboratory
2121 Paularino
Newport Beach, California 92663
Attn: Technical Librarian
Designee: George Moc
Director, Research

Dynamic Science Corporation
1445 Huntington Drive
South Pasadena, California
Attn: M. Gerstein

General Dynamics/Astronautics
Library & Information Services (128-00)
P.O. Box 1128
San Diego, California 92112
Attn: Technical Librarian
Designee: Frank Dore

General Electric Company
Advanced Engine & Technology Dept.
Cincinnati, Ohio 45215
Attn: Technical Librarian
Designee: D. Suichu

General Electric Company
Malta Test Station
Ballston Spa, New York
Attn: Alfred Graham, Manager
Rocket Engines

General Electric Company
Re-Entry Systems Department
3198 Chestnut Street
Philadelphia, Pennsylvania 19101
Attn: Technical Librarian
Designee: F.E. Schultz

Geophysics Corporation of America
Technical Division
Bedford, Massachusetts
Attn: A.C. Toby

Grumman Aircraft Engineering Corp.
Bethpage
Long Island, New York
Attn: Technical Librarian
Designee: Joseph Gavin

Ling-Temco-Vought Corporation
Astronautics
P.O. Box 5907
Dallas, Texas 75222
Attn: Technical Librarian
Designee: Warren C. Trent

Arthur D. Little, Inc.
20 Acorn Park
Cambridge, Massachusetts 02140
Attn: E. Karl Bastress
Attn: Technical Librarian

Lockheed Missiles & Space Co.
P.O. Box 504
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Attn: Technical Information Center
Designee: Y.C. Lee

Lockheed Propulsion Company
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Designee: H.L. Thackwell

McDonnell Aircraft Corporation
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Attn: Technical Librarian
Designee: R.A. Herzmark

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Designee: Warren P. Boardman, Jr.

Martin Marietta Corporation
Denver Division
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Denver, Colorado 80201
Attn: Technical Librarian
Designee: J.D. Goodlette (A-241)

Multi-Tech. Inc.
Box 4186 No. Annex
San Fernando, California
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Northrup Space Laboratories
3401 West Broadway
Hawthorne, California
Attn: Technical Librarian
Designee: William Howard

Rocket Research Corporation
520 South Portland Street
Seattle, Washington 98108
Attn: Technical Librarian
Designee: Foy McCullough, Jr.

Rocketdyne
Division of North American Aviation
6633 Canoga Avenue
Canoga Park, California 91304
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Designee: E.B. Monteath

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Rohm & Haas Company
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Stanford Research Institute
333 Ravenswood Avenue
Menlo Park, California 94025
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Thiokol Chemical Corporation
Huntsville Division
Huntsville, Alabama
Attn: Technical Librarian
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TRW Systems
One Space Park
Redondo Beach, California 90278
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Pratt & Whitney Aircraft Company
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567 Main Street
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